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Einstein Warped G_2 and Spin(7) Manifolds

Víctor Manero, Luis Ugarte

Departamento de Matemáticas - I.U.M.A., Universidad de Zaragoza, Campus Plaza San Francisco, 50009 Zaragoza, Spain. E-mail: vmanero@unizar.es; ugarte@unizar.es

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Abstract: In this paper most of the classes of G_2-structures with Einstein induced metric of negative, null, or positive scalar curvature are realized. This is carried out by means of warped G_2-structures with fiber an Einstein SU(3) manifold. The torsion forms of any warped G_2-structure are explicitly described in terms of the torsion forms of the SU(3)-structure and the warping function, which allows to give characterizations of the principal classes of Einstein warped G_2 manifolds. Similar results are obtained for Einstein warped Spin(7) manifolds with fiber a G_2 manifold.

Introduction

The relation between geometric structures (such as almost Hermitian or G_2-structures, among others) and Einstein metrics has been deeply studied by many different authors. In particular, one of the most important problems related with this issue is the longstanding conjecture due to Goldberg [26]:

"*A compact almost Kähler Einstein manifold is Kähler*".

Partial affirmative answers have been obtained under some additional curvature conditions. For instance, in [42] Sekigawa proved that assuming non-negative scalar curvature the conjecture is true. However, the general case is still open. Concerning the non-compact version of this conjecture, Apostolov, Draghici and Moroianu found a counterexample which is described in [2]. This example consists on a non-compact solvmanifold (solvable Lie group) endowed with a left-invariant almost Kähler structure whose induced metric is Einstein. As the almost complex structure is not integrable, the almost Kähler structure is not Kähler.

A G_2-structure on a 7-dimensional manifold M consists of a reduction of the structure group of its frame bundle to the Lie group G_2. Equivalently, such structure can be characterized by the existence of a global non-degenerate 3-form φ on M. Any G_2-structure has an induced Riemannian metric g_φ. When $d\varphi = 0$ the manifold (M, φ) is called closed G_2 manifold, and if in addition the 3-form φ is coclosed then it is necessarily parallel with respect to the Levi-Civita connection of g_φ [20]. Parallel G_2 manifolds are Ricci flat and have holonomy in G_2. Gibbons, Page and Pope described a G_2-analogue of the Goldberg conjecture in [27] where they studied supersymmetric string solutions on closed G_2-manifolds. This analogue can be stated as follows:

"*A compact Einstein closed* G_2 *manifold is parallel*".

In [13] Cleyton and Ivanov answer positively to this question. For the non-compact version, several authors have given partial affirmative answers under some additional conditions. For example, in [9] it is shown that every Einstein closed G_2 manifold with non-negative scalar curvature is parallel. In [15] the authors proved that Einstein closed G_2-manifolds which are also $*$-Einstein are, in fact, parallel. In [19] it is shown that in contrast to the almost Kähler case, a seven-dimensional solvmanifold cannot admit any left-invariant closed G_2-structure such that its induced metric is Einstein, unless it is parallel.

Up to this point, a question that naturally arises is the following: which classes of G_2-structures can induce an Einstein metric? Our goal in this paper is to show that one can realize most of the classes of G_2-structures with Einstein induced metric of negative, null or positive scalar curvature (see Table 5 and Theorem 5.2). We also study the analogous problem for Spin(7) manifolds (see Table 6 and Theorem 7.7). For the construction of such structures, we will consider Einstein warped G_2, resp. Spin(7), manifolds with fiber an Einstein SU(3), resp. G_2 manifold. Next we explain in more detail the contents of the paper.

In Sect. 1 we recall some well known results about SU(3)-structures (ω, ψ_+) on a 6-dimensional manifold L, such as the description of the scalar curvature of the induced metric g_{ω,ψ_+} and the principal classes of SU(3)-structures in terms of their torsion forms [4,16]. Section 2 is devoted to general results about G_2-structures φ on a 7-dimensional manifold M following [9,16]. We also recall the sixteen Fernández-Gray G_2-classes $\mathcal{P}$, $\mathcal{X}_i$, $\mathcal{X}_i \oplus \mathcal{X}_j$, $\mathcal{X}_i \oplus \mathcal{X}_j \oplus \mathcal{X}_k$ and $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$, as well as their description in terms of the torsion forms τ_0, τ_1, τ_2, τ_3 of the G_2-structure. In Sect. 3, a class of G_2-structures on warped products $M = I_f \times L$ with fiber an SU(3) manifold L is considered, which provides a natural extension of the well-known usual, exponential and sine cones (see Proposition 3.1). Different constructions of G-structures based on warped products or cones have been studied by many authors (see for instance [1,3,5–7,15,21,22] and the references therein). We obtain in Theorem 3.4 an explicit description of the torsion forms of the warped G_2-structure in terms of the torsion forms of the SU(3)-structure and the warping function f.

Our goal in Sect. 4 is to construct Einstein 7-manifolds in the different G_2-classes by means of warped products of certain Einstein SU(3) manifolds. In this way explicit Einstein examples with scalar curvatures of different signs are obtained. In Sect. 4.1 we focus on the principal classes of G_2 manifolds, giving characterizations for the existence of a parallel, nearly parallel or Einstein locally conformal parallel warped G_2-structure in terms of the SU(3) geometry of the fiber. Such G_2-structures correspond to the classes $\mathcal{P}$, $\mathcal{X}_1$ and $\mathcal{X}_4$, respectively. For the G_2-class $\mathcal{X}_2 \oplus \mathcal{X}_3$ it is proved that if a warped G_2 manifold M is Einstein then it is parallel (see Proposition 4.6), in particular the G_2-analogue of the Goldberg conjecture holds for warped G_2 manifolds, as closed G_2 manifolds constitute the class $\mathcal{X}_2$.

In Sect. 4.2 we obtain Einstein coclosed G_2-structures, i.e. in the class $\mathcal{X}_1 \oplus \mathcal{X}_3$, on warped products of SU(3) manifolds of type $\mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_3$, and apply the construction to the manifold $S^3 \times S^3$ endowed with one of the SU(3)-structures found in [41]. In Sect. 4.3 we construct Einstein G_2 manifolds in different classes starting with a 6-manifold endowed with a coupled structure. Coupled SU(3)-structures were first introduced in [40] and have torsion class $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$, so they are half-flat and generalize the nearly Kähler structures. The twistor space $\mathcal{Z}$ over a self-dual Einstein 4-manifold has an Einstein coupled SU(3)-structure [43], which is used in [22] to construct a Ricci-flat locally conformal closed G_2 manifold, i.e. in the class $\mathcal{X}_2 \oplus \mathcal{X}_4$ (see [23] for Einstein solvmanifolds in this class with negative scalar curvature). In Theorems 4.14, 4.17 and 4.18we construct Einstein G_2 manifolds of negative, null and positive scalar curvature in the classes $\mathcal{X}_2 \oplus \mathcal{X}_4$, $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$, $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ and $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$. An Einstein 6-solvmanifold S, of negative scalar curvature, is considered in Sect. 4.4 to obtain an Einstein G_2 manifold on the hyperbolic cosine cone over S.

Motivated by the classification problem studied in [12], in Sect. 5 we realize most of the G_2-classes in the Einstein setting with scalar curvature of different signs (see Theorem 5.2). More concretely, in the Ricci flat case and in the case of positive scalar curvature, there exist Einstein warped G_2-structures of every admissible strict type, except possibly for $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$. On the other hand, there are Einstein warped G_2-structures with negative scalar curvature of every admissible strict type, except for $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{X}_2 \oplus \mathcal{X}_3$, and possibly for $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$. Table 5 shows concrete Einstein examples, when they exist, in the different G_2-classes together with information on the SU(3) geometry of the fibers. At the end of Sect. 5, explicit families of Einstein G_2-structures with identical Riemannian metric but having different G_2 type are given (see [1,9,28, 34,36] for related results).

Section 6 is devoted to warped Spin(7) manifolds $(N = I_f \times M, \phi)$ with fiber a G_2 manifold (M, φ). In Theorem 6.3 we describe the torsion forms λ_1, λ_5 of the Spin(7)-structure ϕ in terms of the torsion forms of the fiber, which allows to give characterizations for the existence of a parallel or an Einstein locally conformal parallel warped Spin(7)-structure in terms of the G_2 geometry of the fiber. In Sect. 7 Einstein 8-manifolds in the different Spin(7)-classes, i.e. $\mathcal{P}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ and the general class $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, are constructed. For zero or positive scalar curvatures, there are Einstein warped Spin(7)-structures of every admissible strict type, whereas for negative scalar curvature there are Einstein warped Spin(7)-structures of every admissible strict type, except for $\mathcal{Y}_2$ (see Theorem 7.7 and Table 6).

1. SU(3)-Structures

An SU(3)-structure on a 6-dimensional manifold L consists of a triple (g, J, Ψ) such that g is a Riemannian metric, J is an almost complex structure compatible with the metric, and Ψ is a complex volume form satisfying

$$\frac{3}{4} i\, \Psi \wedge \overline{\Psi} = \omega^3,$$

where ω is the fundamental form associated to the almost Hermitian structure (g, J). Note that an SU(3)-structure on a 6-dimensional manifold L can be described by the pair (ω, ψ_+), where ψ_+ is the real part of the complex volume form Ψ. Indeed, ψ_+ determines the almost complex structure J, and the imaginary part ψ_- of the form Ψ satisfies $\psi_- = J\psi_+$ (see [29]). We will denote by g_{ω,ψ_+} the Riemannian metric induced by the SU(3)-structure.

As it is described in [4], the intrinsic torsion of an SU(3)-structure can be given in terms of the derivatives of the forms ω, ψ_+ and ψ_-. Consider the natural action of the group SU(3) on the spaces $\Omega^p(L)$ of differential p-forms on L, and more concretely, the SU(3) irreducible subspaces of $\Omega^2(L)$ and $\Omega^3(L)$. One has the following decompositions [4,16]:

$$\Omega^2(L) = \Omega^2_1(L) \oplus \Omega^2_6(L) \oplus \Omega^2_8(L),$$

where

$$\begin{aligned}
\Omega^2_1(L) &= \{f\,\omega \mid f \in \mathcal{C}^\infty(L)\},\\
\Omega^2_6(L) &= \{*_6 J(\alpha \wedge \psi_+) \mid \alpha \in \Omega^1(L)\} = \{\beta \in \Omega^2(L) \mid J\beta = -\beta\},\\
\Omega^2_8(L) &= \{\beta \in \Omega^2(L) \mid \beta \wedge \psi_+ = 0,\ *_6 J\beta = -\beta \wedge \omega\}\\
&= \{\beta \in \Omega^2(L) \mid J\beta = \beta,\ \beta \wedge \omega^2 = 0\},
\end{aligned}$$

and

$$\Omega^3(L) = \Omega^3_{1_+}(L) \oplus \Omega^3_{1_-}(L) \oplus \Omega^3_6(L) \oplus \Omega^3_{12}(L)$$

with

$$\begin{aligned}
\Omega^3_{1_\pm}(L) &= \{f\,\psi_\pm \mid f \in \mathcal{C}^\infty(L)\},\\
\Omega^3_6(L) &= \{\alpha \wedge \omega \mid \alpha \in \Omega^1(L)\} = \{\gamma \in \Omega^3(L) \mid *_6 J\gamma = \gamma\},\\
\Omega^3_{12}(L) &= \{\gamma \in \Omega^3(L) \mid \gamma \wedge \omega = 0,\ \gamma \wedge \psi_\pm = 0\}.
\end{aligned}$$

Here, $*_6$ denotes the Hodge star operator, and $\Omega^p_k(L)$ is the SU(3) irreducible space of p-forms of dimension k at every point. The decomposition on the other degrees is obtained via the isomorphism described by the Hodge star operator $*_6$, i.e. $*_6\, \Omega^p_k(L) \cong \Omega^{6-p}_k(L)$.

Thus, the differentials of ω, ψ_+ and ψ_- can be decomposed into summands belonging to the SU(3) invariant spaces as follows:

$$\begin{aligned}
d\omega &= -\frac{3}{2}\sigma_0\, \psi_+ + \frac{3}{2}\pi_0\, \psi_- + \nu_1 \wedge \omega + \nu_3,\\
d\psi_+ &= \pi_0\, \omega^2 + \pi_1 \wedge \psi_+ - \pi_2 \wedge \omega,\\
d\psi_- &= \sigma_0\, \omega^2 + \pi_1 \wedge \psi_- - \sigma_2 \wedge \omega,
\end{aligned}\tag{1}$$

where $\sigma_0, \pi_0 \in \mathcal{C}^\infty(L)$, $\pi_1, \nu_1 \in \Omega^1(L)$, $\pi_2, \sigma_2 \in \Omega^2_8(L)$ and $\nu_3 \in \Omega^3_{12}(L)$ are called the *torsion forms*. Note that in the last equality, $\pi_1 \wedge \psi_- = J\pi_1 \wedge \psi_+$ accordingly to [4].

Bedulli and Vezzoni derived the Ricci tensor of the metric g_{ω,ψ_+} induced by the SU(3)-structure in terms of the torsion forms. In [4, Theorem 3.4], they find the following expression for the scalar curvature:

$$\begin{aligned}
Scal(g_{\omega,\psi_+}) = {} & \frac{15}{2}\pi_0^2 + \frac{15}{2}\sigma_0^2 + 2d^{*_6}\pi_1 + 2d^{*_6}\nu_1 - |\nu_1|^2 - \frac{1}{2}|\sigma_2|^2\\
& -\frac{1}{2}|\pi_2|^2 - \frac{1}{2}|\nu_3|^2 + 4\langle \pi_1, \nu_1\rangle.
\end{aligned}\tag{2}$$

Here, d^{*_6} denotes the codifferential, i.e. the adjoint of the exterior derivative with respect to the metric.

As it is described in [16] the torsion of an SU(3)-structure, namely T, lies in the space

$$T \in \mathcal{W}_1^\pm \oplus \mathcal{W}_2^\pm \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

where $\mathcal{W}_i$ are the irreducible components under the action of the group SU(3). The spaces $\mathcal{W}_i$ are related to the torsion forms by Table 1.

Table 1. Principal classes of SU(3)-structures

Class	Non-zero torsion form
$\{0\}$	–
$\mathcal{W}_1^+$	π_0
$\mathcal{W}_1^-$	σ_0
$\mathcal{W}_2^+$	π_2
$\mathcal{W}_2^-$	σ_2
$\mathcal{W}_3$	ν_3
$\mathcal{W}_4$	ν_1
$\mathcal{W}_5$	π_1

Hence, torsion forms provide a useful tool to describe the principal classes of SU(3)-structures. For instance, SU(3)-structures with zero torsion are called integrable, or Calabi-Yau, their holonomy is contained in SU(3) and they are Ricci flat. The SU(3)-structures in the class $\mathcal{W}_1^-$ are nearly Kähler. They are Einstein and all the torsion forms vanish except for σ_0. There are only finitely many homogeneous nearly Kähler manifolds [11] and new complete inhomogeneous examples on S^6 and $S^3 \times S^3$ are found recently in [24]. Other well known SU(3)-structures are the half-flat structures, for which $\pi_0 = \pi_1 = \nu_1 = \pi_2 = 0$, and the nearly half-flat structures, characterized by $\pi_1 = \nu_1 = \sigma_2 = 0$. Half-flat structures were first considered in [30] (see also [16]) and the class of nearly half-flat structures was introduced in [21], and these structures can be evolved to a parallel and to a nearly parallel G_2-structure, respectively.

In this paper the SU(3)-structures in the classes $\mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_3$ and $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ will play a role in the construction of Einstein G_2 manifolds (see Sects. 4.2 and 4.3). The structures in the first class are characterized by $\pi_1 = \nu_1 = \pi_2 = \sigma_2 = 0$, and the structures in the second class are known as coupled SU(3)-structures. Coupled SU(3)-structures were first introduced in [40] (see also [22]) and they are characterized by the condition $d\omega = -\frac{3}{2}\sigma_0\, \psi_+$, where σ_0 is constant, which is equivalent to the vanishing of all the torsion forms except σ_0 and σ_2. Thus, coupled structures are half-flat and they generalize the nearly Kähler structures.

We end this section recalling some well-known identities concerning SU(3)-structures that will be useful in the next sections.

Lemma 1.1. *Consider an* SU(3)*-structure* (ω, ψ_+, ψ_-) *on a* 6*-manifold* L*. Then, for any* 1*-form* $\tau \in \Omega^1(L)$ *the following identities hold:*

- $*_6(\tau \wedge \omega) \wedge \omega = *_6(\tau \wedge \psi_+) \wedge \psi_+ = *_6(\tau \wedge \psi_-) \wedge \psi_- = 2 *_6 \tau,$
- $*_6(\tau \wedge \psi_+) \wedge \psi_- = - *_6 (\tau \wedge \psi_-) \wedge \psi_+ = -\tau \wedge \omega^2.$

Proof. Let $\{e^1, \dots, e^6\}$ be a basis adapted to the SU(3)-structure, i.e. a local orthonormal basis such that the forms ω, ψ_+ and ψ_- have the following expressions

$$\omega = e^{12} + e^{34} + e^{56}, \qquad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$
$$\psi_- = e^{136} + e^{145} + e^{235} - e^{246}.$$

Here we denote by e^{ij}, resp. e^{ijk}, the wedge product $e^i \wedge e^j$, resp. $e^i \wedge e^j \wedge e^k$. Now, a generic 1-form on L can be written locally as $\tau = \sum_{i=1}^{7} a_i e^i$, with $a_i \in \mathcal{C}^\infty(L)$, and the result follows by a direct calculation. $\square$

2. G_2-Structures

A G_2-structure on a 7-dimensional manifold M consists of a reduction of the structure group of its frame bundle to the Lie group G_2. Equivalently, the existence of such structure can be characterized by the existence of a global non-degenerate 3-form φ on M which can be locally written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{3}$$

where $\{e^1, \ldots, e^7\}$ is a local basis of 1-forms on M. The presence of a G_2-structure φ on a manifold defines a volume form vol_7 and a Riemannian metric g_φ which satisfy

$$g_\varphi(X, Y) vol_7 = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

for every X, Y vector fields on M.

Let (M, φ) be a G_2 manifold. Then, the group G_2 acts on the space $\Omega^p(M)$ of differential p-forms on the manifold M. This action is irreducible on $\Omega^1(M)$ and $\Omega^6(M)$, but it is reducible for $\Omega^p(M)$ with $2 \le p \le 5$. Since the Hodge star operator $*_7$ induces an isomorphism between the spaces of p-forms and $(7-p)$-forms on M, we only need to describe the decompositions for $p = 2$ and 3. In [9] it is shown that the G_2 irreducible decompositions for $p = 2$ and 3 are

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M),$$

where

$$\Omega^2_7(M) = \{*_7(\alpha \wedge *_7\varphi) \mid \alpha \in \Omega^1(M)\},$$
$$\Omega^2_{14}(M) = \{\beta \in \Omega^2(M) \mid \beta \wedge \varphi = - *_7 \beta\} = \{\beta \in \Omega^2(M) \mid \beta \wedge *_7\varphi = 0\},$$

and

$$\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M),$$

with

$$\Omega^3_1(M) = \{f\varphi \mid f \in \mathcal{C}^\infty(M)\},$$
$$\Omega^3_7(M) = \{*_7(\alpha \wedge \varphi) \mid \alpha \in \Omega^1(M)\},$$
$$\Omega^3_{27}(M) = \{\gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0,\ \gamma \wedge *_7\varphi = 0\},$$

where $\Omega^p_k(M)$ denotes a G_2 irreducible space of p-forms of dimension k at every point. Note that the description on the other degrees are obtained via the isomorphism described by the Hodge star operator, i.e. $*_7\, \Omega^p_k(M) \cong \Omega^{7-p}_k(M)$.

As it is pointed out in [9], it is useful to recognize the scaling factors that the isomorphisms between these G_2 irreducible spaces introduce. For example, for any $\kappa \in \Omega^1(M)$ one has

$$\begin{aligned} *_7 \left(*_7 (\kappa \wedge \varphi) \wedge \varphi \right) &= -4\kappa, \\ *_7 \left(*_7 (\kappa \wedge *_7\varphi) \wedge *_7\varphi \right) &= \ 3\kappa. \end{aligned} \tag{4}$$

The G_2 type decomposition of forms on M allows to express the exterior derivative of φ and $*_7\varphi$ as follows

$$\begin{aligned} d\varphi &= \tau_0 *_7\varphi + 3\, \tau_1 \wedge \varphi + *_7\, \tau_3, \\ d *_7\varphi &= 4\, \tau_1 \wedge *_7\varphi + \tau_2 \wedge \varphi, \end{aligned} \tag{5}$$

Table 2. Principal classes of G_2-structures

Class	Torsion forms	Structure
$\mathcal{P}$	$\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$	Parallel
$\mathcal{X}_1$	$\tau_1 = \tau_2 = \tau_3 = 0$	Nearly parallel
$\mathcal{X}_2$	$\tau_0 = \tau_1 = \tau_3 = 0$	Closed
$\mathcal{X}_3$	$\tau_0 = \tau_1 = \tau_2 = 0$	Coclosed of pure type
$\mathcal{X}_4$	$\tau_0 = \tau_2 = \tau_3 = 0$	Locally conformal parallel
$\mathcal{X}_1 \oplus \mathcal{X}_3$	$\tau_1 = \tau_2 = 0$	Coclosed

where $\tau_0 \in \mathcal{C}^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$ are called the *torsion forms* of the G_2-structure.

According to [20], the covariant derivative of φ can be decomposed into four irreducible components, namely X_1, X_2, X_3 and X_4. Thus, a G_2-structure is said to be of type $\mathcal{P}$, $\mathcal{X}_i$, $\mathcal{X}_i \oplus \mathcal{X}_j$, $\mathcal{X}_i \oplus \mathcal{X}_j \oplus \mathcal{X}_k$ or $\mathcal{X}$ if the covariant derivative $\nabla^{g_\varphi}\varphi$ lies in $\{0\}$, X_i, $X_i \oplus X_j$, $X_i \oplus X_j \oplus X_k$ or $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$, respectively. Hence, there exist 16 different classes of G_2-structures. These classes can be described in terms of the behavior of the torsion forms τ_0, τ_1, τ_2, τ_3 [16]. In Table 2 the principal Fernández-Gray classes of G_2-structures are given.

Hence, torsion forms constitute a useful tool to describe different G_2-structures. Moreover, as it was shown by Bryant in [9], one can also describe the scalar curvature of a G_2 manifold in terms of its torsion forms by

$$Scal(g_\varphi) = 12\, d^{*_7}\tau_1 + \frac{21}{8}\,\tau_0^2 + 30\,|\tau_1|^2 - \frac{1}{2}\,|\tau_2|^2 - \frac{1}{2}\,|\tau_3|^2, \tag{6}$$

where d^{*_7} is the codifferential with respect to the metric g_φ on M.

The geometry of G_2-structures in the different classes above has been studied by many authors. Parallel G_2 manifolds have holonomy in G_2 and they are Ricci-flat. Examples of manifolds with G_2 holonomy are constructed in [8,10,32]. On the other hand, any (strict) nearly parallel G_2 manifold is Einstein with positive scalar curvature [25]. The classification of G_2 manifolds, initiated in [20], was completed in [12] both in the noncompact and compact cases. In Sect. 5 we realize most of the G_2-classes in the Einstein setting with scalar curvature of different signs.

3. Warped G_2-Structures

In this section we consider a class of G_2-structures on warped products with fiber an SU(3) manifold, and we obtain an explicit description of the torsion forms of the warped G_2-structure in terms of the torsion forms of the SU(3)-structure.

The presence of an SU(3)-structure on a 6-dimensional manifold provides a way to obtain 7-dimensional manifolds endowed with G_2-structures. Indeed, consider L a 6-dimensional manifold endowed with an SU(3)-structure (ω, ψ_+, ψ_-). Let M be the Riemannian product $M = \mathbb{R} \times L$, and denote by

$$p\colon M \longrightarrow \mathbb{R}, \qquad q\colon M \longrightarrow L,$$

the projections. Then, the 3-form

$$\varphi = q^*(\omega) \wedge p^*(dt) + q^*(\psi_+),$$

where t is the coordinate on $\mathbb{R}$, defines a G_2-structure on M. In the following, we will identify ω, ψ_+ and ψ_- with their pullbacks onto M.

We will consider a slightly more general class of G_2-structures given by the warped product construction. Let (B, g_B) and (F, g_F) be two Riemannian manifolds, and let f be a nowhere vanishing smooth function on B. In this paper we suppose that f is never a constant function. Denote by p and q the projections of $B \times F$ onto B and F, respectively. Recall that the warped product, namely $M = B \times_f F$, is the product manifold $B \times F$ endowed with the metric g given by

$$g = f^2\, q^*(g_F) + p^*(g_B).$$

The manifold B is called the base of M, F the fiber, and the warped product is called trivial if f is a constant function.

In what follows, we consider $F = L$ and a 1-dimensional base B. More concretely, $B = I_f \subset \mathbb{R}$ is an open interval where the function $f(t)$ does not vanish. In the next result we introduce the class of G_2-structures that will be studied.

Proposition 3.1. *Let $(L, \omega, \psi_+, \psi_-)$ be an* SU(3) *manifold and consider functions $f, \alpha, \beta\colon I_f \longrightarrow \mathbb{R}$, with $\alpha^2(t) + \beta^2(t) = 1$. Then, the form on $M = I_f \times L$ given by*

$$\varphi = f^2(t)\, \omega \wedge dt + f^3(t)\big(\alpha(t)\psi_+ - \beta(t)\psi_-\big) \tag{7}$$

defines a family of G_2*-structures whose induced metric is*

$$g_\varphi = f^2(t)\, g_{\omega,\psi_+} + dt^2.$$

Proof. Consider $\{e^1, \dots, e^6\}$ a local orthonormal basis of 1-forms for which the SU(3)-structure has its canonical expression. Then, with respect to the basis

$$\{h^1, \dots, h^7\} = \{f(t)e^1, \dots, f(t)e^4, f(t)\big(\alpha(t)e^5 - \beta(t)e^6\big), f(t)\big(\beta(t)e^5 + \alpha(t)e^6\big), dt\}$$

the 3-form φ can be written as in (3), and therefore $\{h^1, \dots, h^7\}$ is a local orthonormal basis for the metric g_φ. Thus,

$$g_\varphi = \sum_{i=1}^{7} h^i \otimes h^i = f^2(t) \sum_{i=1}^{6} e^i \otimes e^i + dt \otimes dt = f^2(t)\, g_{\omega,\psi_+} + dt^2.$$

□

It is worthy to remark that in the previous proposition we have enlarged the set of G_2-structures φ, inducing the same metric g_φ, by using functions $\alpha(t)$ and $\beta(t)$ due to the phase freedom for the (3,0)-form of the SU(3)-structure. This will allow us to obtain Einstein metrics that could not be found with α and β constant.

According to Proposition 3.1, if $(L, \omega, \psi_+, \psi_-)$ is an SU(3) manifold, then the G_2 manifold $M = I_f \times L$ with φ described in (7) is precisely the warped product manifold $M = I_f \times_f L$. In what follows, any such G_2-structure φ will be called *warped* G_2-*structure*, and we will refer to the pair $(M = I_f \times L, \varphi)$ as a *warped* G_2 *manifold*. Notice that the warped G_2-structure generalizes the well-known ideas of cone and sine-cone that appear in the literature.

Next we will obtain an explicit description of the torsion forms of the warped G_2-structure on $M = I_f \times L$ in terms of the torsion forms of the SU(3)-structure on L, the warping function f, and the functions α, β. For the sake of simplicity, in the next results we will not write the t-dependence of the functions f, α and β.

The following lemma will be useful to relate the Hodge star operators $*_6$ and $*_7$ induced by the SU(3) and G_2 structures, respectively.

Lemma 3.2. *Let $\gamma \in \Omega^p(L)$ be a differential p-form on L, and let $*_6$ and $*_7$ be the Hodge star operators induced by the structures (ω, ψ_+, ψ_-) and φ, respectively. Then,*

$$*_7\gamma = f^{6-2p} *_6 \gamma \wedge dt, \qquad *_7(\gamma \wedge dt) = (-1)^p f^{6-2p} *_6 \gamma.$$

Proof. It is an immediate consequence of the definition of the Hodge star operator and the fact that $*_6$ and $*_7$ are determined, respectively, by $(g_{\omega,\psi_+}, vol_6 = \frac{1}{6}\omega^3)$ and (g_φ, vol_7), with $vol_7 = f^6 vol_6 \wedge dt$. □

Proposition 3.3. *Let φ be a warped G_2-structure on $M = I_f \times L$. Then,*

$$\begin{aligned}
d\varphi = &-f^2\Big(\frac{3}{2}\sigma_0 + 3f'\alpha + f\alpha'\Big)\psi_+ \wedge dt + f^2\Big(\frac{3}{2}\pi_0 + 3f'\beta + f\beta'\Big)\psi_- \wedge dt\\
&+ f^3\big(\alpha\,\pi_0 - \beta\,\sigma_0\big)\omega^2 + f^2\nu_1 \wedge \omega \wedge dt + f^2\nu_3 \wedge dt\\
&+ f^3\pi_1 \wedge (\alpha\,\psi_+ - \beta\,\psi_-) - f^3\big(\alpha\,\pi_2 - \beta\,\sigma_2\big) \wedge \omega,\\
d *_7 \varphi = \; & f^3\big(2f' + \beta\,\pi_0 + \alpha\,\sigma_0\big)\omega^2 \wedge dt + f^4\nu_1 \wedge \omega^2\\
&+ f^3\pi_1 \wedge \big(\beta\,\psi_+ + \alpha\,\psi_-\big) \wedge dt - f^3\big(\beta\,\pi_2 + \alpha\,\sigma_2\big) \wedge \omega \wedge dt,
\end{aligned}$$

where we denote by $\pi_0, \sigma_0, \pi_1, \nu_1, \pi_2, \sigma_2$ and ν_3 the torsion forms of the SU(3)*-structure (ω, ψ_+, ψ_+) on L.*

Proof. For $d\varphi$, the result is a direct consequence of Eq. (1) and Proposition 3.1. On the other hand, from Lemma 3.2 it follows that

$$*_7\varphi = \frac{1}{2}f^4\,\omega \wedge \omega + f^3\big(\beta\,\psi_+ + \alpha\,\psi_-\big) \wedge dt,$$

and the result for $d *_7 \varphi$ is obtained also as a direct consequence of (1) and Proposition 3.1. □

Theorem 3.4. *Let $(L, \omega, \psi_+, \psi_-)$ be an* SU(3) *manifold with torsion forms $\pi_0, \sigma_0, \pi_1, \nu_1, \pi_2, \sigma_2$ and ν_3. Then, the torsion forms of a warped* G_2 *manifold $(M = I_f \times L, \varphi)$ are given by*

$$\begin{aligned}
\tau_0 &= \frac{4}{7f}\big(3\,\pi_0\,\alpha - 3\,\sigma_0\,\beta + f\alpha\beta' - f\beta\alpha'\big),\\
\tau_1 &= \frac{1}{2f}\big(\pi_0\,\beta + \sigma_0\,\alpha + 2f'\big)dt + \frac{\nu_1}{6} + \frac{\pi_1}{6},\\
\tau_2 &= -\frac{2}{3} *_6(\nu_1 \wedge \omega^2) \wedge dt + \frac{1}{3} *_6(\pi_1 \wedge \omega^2) \wedge dt\\
&\quad -\frac{1}{3}f\beta *_6(\pi_1 \wedge \psi_+) - \frac{1}{3}f\alpha *_6(\pi_1 \wedge \psi_-)\\
&\quad +\frac{2}{3}f\beta *_6(\nu_1 \wedge \psi_+) + \frac{2}{3}f\alpha *_6(\nu_1 \wedge \psi_-) - f\beta\,\pi_2 - f\alpha\,\sigma_2,\\
\tau_3 &= -\frac{3}{14}f^2\left(\pi_0\,\alpha^2 - \sigma_0\,\alpha\beta - 2f\beta'\right)\psi_+ + \frac{3}{14}f^2\left(\pi_0\,\alpha\beta - \sigma_0\,\beta^2 + 2f\alpha'\right)\psi_-\\
&\quad +\frac{2}{7}f\left(\pi_0\,\alpha - \sigma_0\,\beta - 2f\alpha\beta' + 2f\beta\alpha'\right)\omega \wedge dt - \frac{1}{2} *_6(\nu_1 \wedge \omega) + \frac{1}{2} *_6(\pi_1 \wedge \omega)
\end{aligned}$$

$$
+\frac{1}{2} f\alpha *_6(\pi_1 \wedge \psi_+) \wedge dt - \frac{1}{2} f\beta *_6(\pi_1 \wedge \psi_-) \wedge dt - \frac{1}{2} f\alpha *_6(\nu_1 \wedge \psi_+) \wedge dt
+\frac{1}{2} f\beta *_6(\nu_1 \wedge \psi_-) \wedge dt + f(\alpha\,\pi_2 - \beta\,\sigma_2) \wedge dt - f^2 *_6 \nu_3.
$$

Proof. From (5) it can be easily obtained that

$$
\begin{aligned}
\tau_0 &= \tfrac{1}{7} *_7(d\,\varphi \wedge \varphi), & \tau_2 &= - *_7\, d *_7 \varphi + 4 *_7(\tau_1 \wedge *_7\varphi),\\
\tau_1 &= -\tfrac{1}{12} *_7(*_7 d\,\varphi \wedge \varphi), & \tau_3 &= *_7 d\varphi - \tau_0\,\varphi - 3 *_7(\tau_1 \wedge \varphi).
\end{aligned}
$$

Let us detail the computations for τ_0. By Proposition 3.3 we have

$$
\begin{aligned}
d\varphi \wedge \varphi = \Big[&- f^2\Big(\frac{3}{2}\sigma_0 + 3f'\alpha + f\alpha'\Big)\psi_+ \wedge dt + f^2\Big(\frac{3}{2}\pi_0 + 3f'\beta + f\beta'\Big)\psi_- \wedge dt\\
&+ f^3\big(\pi_0\,\alpha - \sigma_0\,\beta\big)\,\omega^2 + f^2\nu_1 \wedge \omega \wedge dt + f^2\nu_3 \wedge dt\\
&+ f^3\pi_1 \wedge (\alpha\,\psi_+ - \beta\,\psi_-) - f^3\big(\alpha\,\pi_2 - \beta\,\sigma_2\big) \wedge \omega\Big]\\
&\wedge \Big[f^2\omega \wedge dt + f^3\big(\alpha\,\psi_+ - \beta\,\psi_-\big)\Big]\\
= {}& f^5(\pi_0\,\alpha - \sigma_0\,\beta)\,\omega^3 \wedge dt + \alpha f^5\Big(\frac{3}{2}\pi_0 + 3f'\beta + f\beta'\Big)\psi_+ \wedge \psi_- \wedge dt\\
&- \beta f^5\Big(\frac{3}{2}\sigma_0 + 3f'\alpha + f\alpha'\Big)\psi_+ \wedge \psi_- \wedge dt\\
= {}& f^5(\pi_0\,\alpha - \sigma_0\,\beta)\,\omega^3 \wedge dt + f^5\big(\pi_0\,\alpha + \frac{2}{3} f\alpha\beta' - \sigma_0\,\beta - \frac{2}{3} f\beta\alpha'\big)\,\omega^3 \wedge dt\\
= {}& f^5\big(2\pi_0\,\alpha - 2\sigma_0\,\beta + \frac{2}{3} f\alpha\beta' - \frac{2}{3} f\beta\alpha'\big)\,\omega^3 \wedge dt.
\end{aligned}
$$

Therefore, using Lemma 3.2 we get

$$
\tau_0 = \frac{1}{7} *_7(d\,\varphi \wedge \varphi) = \frac{4}{7f}\big(3\pi_0\,\alpha - 3\sigma_0\,\beta + f\alpha\beta' - f\beta\alpha'\big).
$$

Similarly, the results for τ_1, τ_2 and τ_3 follow as a long but standard computation taking into account Proposition 3.3 and Lemmas 1.1 and 3.2. □

An immediate consequence of the previous theorem is the following

Corollary 3.5. *The torsion forms of a warped* G_2*-structure satisfy:*

$$
\begin{aligned}
\tau_0 = 0 &\iff \big\{ i) \quad 3\pi_0\,\alpha - 3\sigma_0\,\beta + f\alpha\beta' - f\beta\alpha' = 0;\\
\tau_1 = 0 &\iff \begin{cases} ii) & \sigma_0\,\alpha + \pi_0\,\beta + 2f' = 0,\\ iii) & \pi_1 = -\nu_1; \end{cases}\\
\tau_2 = 0 &\iff \begin{cases} iv) & \pi_1 = 2\nu_1,\\ v) & \beta\pi_2 + \alpha\sigma_2 = 0; \end{cases}\\
\tau_3 = 0 &\iff \begin{cases} vi) & \pi_0\,\alpha - \sigma_0\,\beta - 2f\alpha\beta' + 2f\beta\alpha' = 0,\\ vii) & \pi_1 = \nu_1,\\ viii) & \alpha\pi_2 - \beta\sigma_2 = 0,\\ ix) & \nu_3 = 0. \end{cases}
\end{aligned}
$$

Proof. The result is obvious for τ_0, τ_1 and τ_2 in view of Theorem 3.4. For τ_3, the vanishing of the first three summands (see Theorem 3.4) is equivalent to $vi)$. Indeed,

$$\pi_0\,\alpha^2 - \sigma_0\,\alpha\beta - 2f\beta' = \alpha\Big(\pi_0\,\alpha - \sigma_0\,\beta - 2f\alpha\beta' + 2f\beta\alpha'\Big)$$

and

$$\pi_0\,\alpha\beta - \sigma_0\,\beta^2 + 2f\alpha' = \beta\Big(\pi_0\,\alpha - \sigma_0\,\beta - 2f\alpha\beta' + 2f\beta\alpha'\Big),$$

where we are using the fact that $\alpha\alpha' = -\beta\beta'$, which follows from the identity $\alpha^2+\beta^2 = 1$. The other conditions $vii)$, $viii)$ and $ix)$ are clear from Theorem 3.4. □

4. Einstein Warped G_2 Manifolds

Our goal in this section is to construct Einstein 7-manifolds in the different G_2-classes by means of warped products of certain Einstein SU(3) manifolds. The G_2-structures are of the form (7), i.e. what we called warped G_2-structures. In this way we will obtain explicit Einstein examples with scalar curvature of different signs. In Sect. 4.1 we study the principal classes of G_2 manifolds, Sect. 4.2 is devoted to coclosed G_2-structures, in Sect. 4.3 warped products of coupled SU(3)-structures are considered, and in Sect. 4.4 we obtain G_2 structures on the hyperbolic cosine cone of Einstein solvmanifolds.

Let us consider the warped product $M = B \times_f F$, i.e. the product manifold $B \times F$ endowed with the metric g given by $g = f^2 q^*(g_F) + p^*(g_B)$, with p and q the projections of $B \times F$ onto B and F, respectively, and f a nowhere vanishing smooth function on B. We denote by Ric^B the lift to M (i.e. the pullback by p) of the Ricci curvature of B, similarly for Ric^F, and let $Hess(f)$ be the lift to M of the Hessian of f. By [38, p. 211] the warped product $M = B \times_f F$ is Einstein with constant λ (i.e. $Ric = \lambda\, g$) if and only if (F, g_F) is Einstein with constant μ (i.e. $Ric^F = \mu\, g_F$) and the following conditions are satisfied:

$$\lambda\, g_B = Ric^B - \frac{d}{f} Hess(f), \qquad \lambda = \frac{\mu}{f^2} - \frac{\Delta f}{f} - (d-1)\left|\frac{\nabla f}{f}\right|^2_{g_B},$$

where $d = \dim F \geq 2$, $\Delta f = \mathrm{tr}\big(Hess(f)\big)$, and ∇f denotes the gradient of f.

Moreover, when the base space B has dimension 1, these equations reduce to

$$(f')^2 + \frac{\lambda}{d} f^2 = \frac{\mu}{d-1}. \tag{8}$$

The behavior of the solutions of (8) depends on the signs of the Einstein constants λ and μ. Nevertheless, up to homotheties, those solutions (besides the constant case) are given in Table 3 (see also [5]).

Table 3. Solutions of the equation (8)

μ	$-(d-1)$	0	$d-1$	$d-1$	$d-1$
λ	$-d$	$-d$	$-d$	0	d
$f(t)$	$\cosh t$	e^t	$\sinh t$	t	$\sin t$

From this table the next result follows

Theorem 4.1 [5, Theorem 9.110]. *Let $M = B \times_f F$ be a warped product, with* $\dim B = 1$ *and* $\dim F = d > 1$. *If M is a complete Einstein manifold, then either M is a Ricci-flat Riemannian product, or $B = \mathbb{R}$, F is Einstein with non-positive scalar curvature and M has negative scalar curvature.*

We consider $B = I_f \subset \mathbb{R}$ an open interval where the function $f(t)$ does not vanish. For the functions in Table 3 we will take generically $I_f = \mathbb{R}$ for $f(t) = \cosh t$ or e^t, $I_f = (0, \infty)$ for $f(t) = \sinh t$ or t, and $I_f = (0, \pi)$ for $f(t) = \sin t$. In the latter case, if F is compact then $g = dt^2 + \sin^2 t\, q^*(g_F)$ defines a metric on the product manifold $[0, \pi] \times F$ with two conical singularities at $t = 0$ and $t = \pi$ (see for instance [6,21]).

In order to use directly Table 3, we will consider the Einstein metric on the fiber F to be "normalized", that is, its Einstein constant is

$$-(d-1), \quad 0, \quad \text{or} \quad d-1,$$

where d denotes the dimension of F, or equivalently, the scalar curvature is

$$-d\,(d-1), \quad 0, \quad \text{or} \quad d\,(d-1),$$

respectively. There is no loss of generality in assuming this condition since every Einstein metric can be normalized via a rescaling. Similar considerations are applied to Einstein metrics on the total space M of the warped product.

4.1. Principal classes of G_2 *manifolds.* In this section we focus on Einstein 7-manifolds in the principal classes of G_2 manifolds, i.e. in the classes $\mathcal{P}$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$. Whereas one can construct Einstein manifolds in the classes $\mathcal{P}$, $\mathcal{X}_1$ and $\mathcal{X}_4$ by means of warped G_2-structures, however we will prove in Proposition 4.6 that such a manifold in the class $\mathcal{X}_2 \oplus \mathcal{X}_3$ is necessarily parallel.

Next, several characterizations will be given for the classes $\mathcal{P}$, $\mathcal{X}_1$ and $\mathcal{X}_4$. We begin with parallel G_2 manifolds.

Proposition 4.2. *There exists a parallel warped* G_2*-structure on $M = I_f \times L$ if and only if the fiber $(L, \omega, \psi_+, \psi_-)$ belongs to $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$ and is Einstein with $Scal(g_{\omega,\psi_+}) = 30$.*

Furthermore, in that case $M = (0, \infty) \times L$ is the t-cone with the G_2*-structure*

$$\varphi = t^2\, \omega \wedge dt + t^3 \left(-\frac{\sigma_0}{2} \psi_+ + \frac{\pi_0}{2} \psi_- \right), \tag{9}$$

where σ_0, π_0 are the (constant) torsion functions of the $\mathrm{SU}(3)$*-structure, which satisfy $\pi_0^2 + \sigma_0^2 = 4$.*

Proof. Let us suppose that the SU(3) manifold $(L, \omega, \psi_+, \psi_-)$ belongs to $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$ and is Einstein with constant 5. Hence, the torsion reduces to π_0 and σ_0, and the Eq. (1) are given by

$$d\omega = -\frac{3}{2}\sigma_0\, \psi_+ + \frac{3}{2}\pi_0\, \psi_-, \qquad d\psi_+ = \pi_0\, \omega^2, \qquad d\psi_- = \sigma_0\, \omega^2.$$

These equations imply that the wedge product of the 1-forms $d\pi_0, d\sigma_0$ by ω^2 is zero, so π_0, σ_0 are constant. Moreover, from (2) we get $30 = Scal(g_{\omega,\psi_+}) = \frac{15}{2}(\pi_0^2 + \sigma_0^2)$, which implies $\pi_0^2+\sigma_0^2=4$. Now, the warped G_2-structure with $f(t)=t, \alpha=-\frac{\sigma_0}{2}$ and $\beta=-\frac{\pi_0}{2}$ satisfies the equations $i)-ix)$ in Corollary 3.5, so it is parallel.

Conversely, let us suppose that there exists a warped G_2-structure that is parallel, i.e. the equations $i)-ix)$ in Corollary 3.5 are satisfied. From *iii)*, *iv)* and *ix)* we have that $\pi_1=\nu_1=\nu_3=0$, and from *v)* and *viii)* we get $\sigma_2=\pi_2=0$ because $\alpha^2(t)+\beta^2(t)=1$. Hence, the manifold (L,ω,ψ_+,ψ_-) belongs to the SU(3)-class $\mathcal{W}_1^+\oplus\mathcal{W}_1^-$, and by the first part of the proof we have that the torsion functions π_0 and σ_0 are constant. Furthermore, by (6) any G_2-parallel structure is Ricci-flat, so from Table 3 we get that the warping function is necessarily $f(t)=t$ and the metric induced by the SU(3)-structure is Einstein with constant $\mu=5$. Notice that (2) implies $\pi_0^2+\sigma_0^2=4$.

Finally, it remains to see that the G_2-structure on the t-cone is given by (9). Let us write $\alpha(t)=\cos\theta(t)$ and $\beta(t)=\sin\theta(t)$, for some function $\theta(t)$. The equations $i)$ and $vi)$ for $f(t)=t$ are equivalent to

$$\pi_0\,\alpha(t)-\sigma_0\,\beta(t)=0,\qquad \theta'(t)=0,$$

which implies that $\alpha(t)$, $\beta(t)$ are constant functions. On the other hand, from the first equation above and the equation $ii)$ for $f(t)=t$, we arrive at the following system

$$\pi_0\,\alpha-\sigma_0\,\beta=0,\quad \sigma_0\,\alpha+\pi_0\,\beta=-2.$$

Now, the condition $\pi_0^2+\sigma_0^2=4$ clearly implies that $\alpha=-\frac{\sigma_0}{2}$ and $\beta=-\frac{\pi_0}{2}$, and the result follows. $\square$

In the following proposition we consider warped G_2 manifolds in the class $\mathcal{X}_1$. The result also gives another characterization of an SU(3) manifold in the class $\mathcal{W}_1^+\oplus\mathcal{W}_1^-$ in terms of a $\sin t$-cone.

Proposition 4.3. *There exists a nearly parallel warped* G_2*-structure on* $M=I_f\times L$ *with* $Scal(g_\varphi)=42$ *if and only if the fiber* (L,ω,ψ_+,ψ_-) *belongs to* $\mathcal{W}_1^+\oplus\mathcal{W}_1^-$ *and is Einstein with* $Scal(g_{\omega,\psi_+})=30$.

Furthermore, in that case $M=(0,\pi)\times L$ *is the* $\sin t$*-cone with the* G_2*-structure*

$$\varphi=\sin^2 t\,\omega\wedge dt+\sin^3 t\,(\cos(\varepsilon\,t+\rho)\,\psi_+-\sin(\varepsilon\,t+\rho)\,\psi_-)\,, \tag{10}$$

where $\varepsilon=\pm1$ *and* ρ *is given in terms of the (constant) torsion functions* σ_0,π_0 *of the* SU(3)*-structure by* $\sigma_0=-2\cos\rho$ *and* $\pi_0=-2\sin\rho$.

Proof. Suppose that the SU(3) manifold belongs to $\mathcal{W}_1^+\oplus\mathcal{W}_1^-$ and is Einstein with constant 5. Hence, the same argument as in the first part of the proof of Proposition 4.2 shows that π_0,σ_0 are constant and $\pi_0^2+\sigma_0^2=4$. Now, the G_2-structure given by (10) satisfies the equations $ii)-ix)$ in Corollary 3.5. Thus, we get a nearly parallel G_2 manifold with Einstein constant equal to 6.

Let us prove the converse. Suppose that there exists a warped product of (L,ω,ψ_+,ψ_-) given by (7) that is a nearly parallel G_2 manifold with Einstein constant 6, i.e. the equations $ii)-ix)$ in Corollary 3.5 are satisfied. The equations *iii)*, *iv)* and *ix)* imply $\pi_1=\nu_1=\nu_3=0$, and from *v)* and *viii)* we get $\sigma_2=\pi_2=0$ because $\alpha^2(t)+\beta^2(t)=1$. On the other hand, by Table 3 we get that the warping function is necessarily $f(t)=\sin t$ and the metric induced by the SU(3)-structure is Einstein with constant $\mu=5$, which implies, by (2), that $\pi_0^2+\sigma_0^2=4$. Hence, the manifold (L,ω,ψ_+,ψ_-) belongs to the SU(3)-class $\mathcal{W}_1^+\oplus\mathcal{W}_1^-$, and the (constant) torsion functions π_0,σ_0 satisfy $\pi_0^2+\sigma_0^2=4$.

It remains to prove that the warped product M must be necessarily the $\sin t$-cone given in (10). To see this, we consider the equations $ii)$ and $vi)$ for $f(t) = \sin t$ in Corollary 3.5. Writing $\alpha(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$, for some function $\theta(t)$, we get

$$\sigma_0\,\alpha(t) + \pi_0\,\beta(t) = -2\,\cos t, \qquad \pi_0\,\alpha(t) - \sigma_0\,\beta(t) = 2\,\theta'(t)\,\sin t.$$

Using $\pi_0^2 + \sigma_0^2 = 4$, we have

$$\alpha(t) = -\frac{1}{2}\,\sigma_0\,\cos t + \frac{1}{2}\,\pi_0\,\theta'(t)\sin t, \qquad \beta(t) = -\frac{1}{2}\,\pi_0\,\cos t - \frac{1}{2}\,\sigma_0\,\theta'(t)\sin t,$$

and from $\alpha^2(t) + \beta^2(t) = 1$ it follows that

$$\left[\left(\theta'(t)\right)^2 - 1\right]\sin^2 t = 0.$$

This implies $\theta'(t) = \pm 1$ and thus $\theta(t) = \varepsilon\, t + \rho$, where $\varepsilon = \pm 1$ and ρ is a constant which, as we show next, it is determined by σ_0 and π_0. Indeed, the equations $ii)$ and $vi)$ are now written as

$$(\sigma_0\,\cos\rho + \pi_0\,\sin\rho + 2)\cos t + \varepsilon(\pi_0\,\cos\rho - \sigma_0\,\sin\rho)\sin t = 0,$$
$$(\pi_0\,\cos\rho - \sigma_0\,\sin\rho)\cos t - \varepsilon(\sigma_0\,\cos\rho + \pi_0\,\sin\rho + 2)\sin t = 0.$$

These equations imply

$$\sigma_0\,\cos\rho + \pi_0\,\sin\rho = -2, \qquad \sigma_0\,\sin\rho - \pi_0\,\cos\rho = 0,$$

whose solution is $\sigma_0 = -2\,\cos\rho$ and $\pi_0 = -2\,\sin\rho$. In conclusion, the G_2-structure is given by (10) and the proof is complete. □

Corollary 4.4. *Let $(L, \omega, \psi_+, \psi_-)$ be an* SU(3) *manifold in $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$ with $Scal(g_{\omega,\psi_+}) =$ 30. Then, the nearly parallel* G_2*-structure on $M = I_f \times L$ given by* (10) *has torsion $\tau_0 = 4\,\varepsilon$ $(\varepsilon = \pm 1)$.*

Proof. It is a direct consequence of Proposition 4.3 and the expression of τ_0 in Theorem 3.4, taking $f(t) = \sin t$, $\alpha(t) = \cos(\varepsilon t + \rho)$, $\beta(t) = \sin(\varepsilon t + \rho)$, $\cos\rho = -\frac{\sigma_0}{2}$ and $\sin\rho = -\frac{\pi_0}{2}$. □

As a consequence of Propositions 4.2 and 4.3 we recover well-known characterizations of a nearly-Kähler manifold L given in [3,21] (see also [7]). Here, and in what follows, we consider that the torsion of a nearly-Kähler manifold is $\sigma_0 = -2$, so the Einstein constant equals 5.

Corollary 4.5. *Let (L, ω, ψ_+) be an* SU(3) *manifold. Then:*

(i) *L is nearly-Kähler if and only if the (usual) cone with the G_2-structure*

$$\varphi = t^2\,\omega \wedge dt + t^3\,\psi_+,$$

is a parallel G_2 manifold;

(ii) *L is nearly-Kähler if and only if the sine-cone with the G_2-structure*

$$\varphi = \sin^2 t\,\omega \wedge dt + \sin^3 t\,(\cos t\,\psi_+ - \sin t\,\psi_-)\,,$$

is a nearly parallel G_2 manifold.

Proof. For (i), just take in (9) the values $\sigma_0 = -2$ and $\pi_0 = 0$. For (ii) we take $\varepsilon = 1$ in (10) and $\rho = 0$, because $-2 = \sigma_0 = -2\cos\rho$ and $0 = \pi_0 = -2\sin\rho$. $\square$

Recall that G_2 manifolds in the class $\mathcal{X}_2 \oplus \mathcal{X}_3$ are characterized in terms of the torsion forms by the conditions $\tau_0 = \tau_1 = 0$.

Proposition 4.6. *A warped G_2 manifold M in the class $\mathcal{X}_2 \oplus \mathcal{X}_3$ is Einstein if and only if it is a parallel G_2 manifold.*

Proof. From Corollary 3.5, if the G_2-structure belongs to the class $\mathcal{X}_2 \oplus \mathcal{X}_3$ then the conditions $i)$, $ii)$ and $iii)$ are satisfied. In addition, an Einstein G_2 manifold with $\tau_0 = \tau_1 = 0$ has non-positive Einstein constant by (6). If such constant is zero then the G_2-structure is parallel. So, in what follows we suppose that the Einstein constant is negative, which after scaling we consider to be -6, and so by Table 3 the possible functions are $f(t) = \cosh t$, e^t, or $\sinh t$. Next we will prove that there is no solution in any of these cases.

From $\alpha^2(t) + \beta^2(t) = 1$ we can write $\alpha(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$, for some real-valued function $\theta(t)$. Thus, $\alpha(t)\beta'(t) - \beta(t)\alpha'(t) = \theta'(t)$, and equations $i)$ and $ii)$ in Corollary 3.5 become:

$i)$ $3\,\pi_0\,\alpha(t) - 3\,\sigma_0\,\beta(t) + \theta'(t)\,f(t) = 0,$
$ii)$ $\sigma_0\,\alpha(t) + \pi_0\,\beta(t) + 2\,f'(t) = 0.$

Multiplying $i)$ by $\alpha(t)$, $ii)$ by $3\beta(t)$, and summing the resulting equations, we get

$$3\,\pi_0 = 3\,\pi_0(\alpha^2(t) + \beta^2(t)) = -\theta'(t)\,\alpha(t)\,f(t) - 6\,\beta(t)\,f'(t).$$

Since π_0 is a function on the fiber manifold L and the right hand side of the equation only depends on t, necessarily there exists a constant C_1 such that

$$\theta'(t)\,\alpha(t)\,f(t) + 6\,\beta(t)\,f'(t) = C_1. \tag{11}$$

Now, multiplying $i)$ by $-\beta(t)$, $ii)$ by $3\alpha(t)$, and summing the resulting equations, we get

$$3\,\sigma_0 = 3\,\sigma_0(\alpha^2(t) + \beta^2(t)) = \theta'(t)\,\beta(t)\,f(t) - 6\,\alpha(t)\,f'(t).$$

Hence, there exists a constant C_2 such that

$$\theta'(t)\,\beta(t)\,f(t) - 6\,\alpha(t)\,f'(t) = C_2. \tag{12}$$

Taking the product of (12) by $\alpha(t)$, the product of (11) by $\beta(t)$, and subtracting the equations, we get $6\,f'(t) = C_1\,\beta(t) - C_2\,\alpha(t)$. In a similar way, taking the product of

(12) by $\beta(t)$, the product of (11) by $\alpha(t)$, and summing the equations, we get $\theta'(t)\, f(t) = C_1\,\alpha(t) + C_2\,\beta(t)$. That is, we arrive at the following system:

$$6\, f'(t) = C_1\,\beta(t) - C_2\,\alpha(t), \tag{13}$$

$$\theta'(t)\, f(t) = C_1\,\alpha(t) + C_2\,\beta(t). \tag{14}$$

Taking the derivative of (14) and using (13) we get $\theta''(t)\, f(t) + \theta'(t)\, f'(t) = C_1\,\alpha'(t) + C_2\,\beta'(t) = -\theta'(t)(C_1\,\beta(t) - C_2\,\alpha(t)) = -6\,\theta'(t)\, f'(t)$, that is

$$\theta''(t)\, f(t) + 7\,\theta'(t)\, f'(t) = 0.$$

Notice that $\theta'(t) = 0$ implies that the functions $\alpha(t)$ and $\beta(t)$ are constant, and then equation $ii)$ cannot be solved for $f(t) = \cosh t$, e^t, or $\sinh t$. Therefore, $\theta'(t) \neq 0$ and we can write the previous equation as

$$\big(\ln\theta'(t) + 7\,\ln f(t)\big)' = 0.$$

Hence, there exists a positive constant C_0 such that

$$\theta'(t) = C_0\, f(t)^{-7}. \tag{15}$$

On the other hand, taking the derivative of (13) and using (14) we get $6\, f''(t) = C_1\,\beta'(t) - C_2\,\alpha'(t) = \theta'(t)(C_1\,\alpha(t) + C_2\,\beta(t)) = (\theta'(t))^2\, f(t)$, that is

$$6\, f''(t) = (\theta'(t))^2\, f(t).$$

Now, using (15), we have $6\, f''(t) = C_0^2\, f(t)^{-13}$, i.e.

$$f(t)^{13} f''(t) = C_0^2/6,$$

which never holds for the functions $f(t) = \cosh t$, e^t, or $\sinh t$. In conclusion, the system $i)$–$iii)$ is never satisfied. $\square$

Since the class $\mathcal{X}_2 \oplus \mathcal{X}_3$ contains the class of closed and the class of coclosed of pure type G_2 manifolds, from Proposition 4.6 we get

Corollary 4.7. *There does not exist any* SU(3) *manifold* $(L, \omega, \psi_+, \psi_-)$ *for which the warped* G_2 *manifold* $M = I_f \times L$ *is Einstein closed or coclosed of pure type, unless it is parallel.*

Remark 4.8. As we recall in the introduction, it is an open question if an Einstein closed G_2 manifold must be parallel. Several authors have proved that this question has an affirmative answer in different particular situations: for compact (and more generally, for $*$-Einstein) manifolds in [13,15], for non-negative scalar curvature in [9], and for solvmanifolds with left invariant G_2-structure in [19]. The corollary above shows that the answer is also affirmative in the class of warped G_2 manifolds.

Now, we turn our attention to Einstein locally conformal parallel G_2 manifolds, i.e. Einstein manifolds in the class $\mathcal{X}_4$.

Proposition 4.9. *There exists an Einstein locally conformal parallel warped* G_2*-structure on* $M = I_f \times L$ *with* $Scal(g_\varphi) = -42$ *if and only the fiber* $(L, \omega, \psi_+, \psi_-)$ *is one of the following:*

- *L is Calabi-Yau, and then $M = \mathbb{R} \times L$ is the exponential-cone with* G_2*-structure $\varphi = e^{2t} \omega \wedge dt + e^{3t} \psi_+$; thus, the unique non-vanishing torsion form is $\tau_1 = dt$.*
- *L belongs to $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$ with $Scal(g_{\omega,\psi_+}) = 30$, and then $M = (0, \infty) \times L$ is the hyperbolic sine-cone with* G_2*-structure $\varphi = \sinh^2 t\, \omega \wedge dt + \sinh^3 t \left(\varepsilon \frac{\sigma_0}{2}\, \psi_+ - \varepsilon \frac{\pi_0}{2}\, \psi_-\right)$, where $\varepsilon = \pm 1$ and σ_0, π_0 are the (constant) torsion functions of the* $SU(3)$*-structure, which satisfy $\pi_0^2 + \sigma_0^2 = 4$. Thus, the non-vanishing torsion form of the warped* G_2*-structure is exactly $\tau_1 = \frac{\varepsilon + \cosh t}{\sinh t} dt$.*

Proof. Suppose there is such a warped product. Using that $\tau_0 = \tau_2 = \tau_3 = 0$ and Corollary 3.5, similarly to the proof of Proposition 4.2 we arrive at the fact that L belongs to $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$, so the torsion reduces to σ_0, π_0. On the other hand, by Theorem 3.4 the unique non-vanishing torsion form of the warped G_2-structure is

$$\tau_1 = \frac{1}{2f}(\pi_0 \beta + \sigma_0 \alpha + 2f')dt. \tag{16}$$

If σ_0, π_0 vanish then L is Calabi-Yau and the warped product is the exponential-cone. If the torsion of L is non-zero then the scalar curvature of L is equal to 30 and $f(t) = \sinh t$. The equations $i)$ and $vi)$ in Corollary 3.5 give the solutions $(\alpha, \beta) = (\varepsilon \frac{\sigma_0}{2}, \varepsilon \frac{\pi_0}{2})$, where $\varepsilon = \pm 1$. Finally, the values of τ_1 for both cases are obtained as a direct consequence of (16). □

Similarly to the previous proposition we have:

Proposition 4.10. *Let $(L, \omega, \psi_+, \psi_-)$ be an* $SU(3)$ *manifold. Then:*

(i) *There exists a Ricci flat locally conformal parallel warped* G_2*-structure on $M = I_f \times L$ if and only if the fiber L belongs to $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$, and then $M = (0, \infty) \times L$ is the cone with* G_2*-structure $\varphi = t^2\, \omega \wedge dt + t^3 \left(\varepsilon \frac{\sigma_0}{2}\, \psi_+ - \varepsilon \frac{\pi_0}{2}\, \psi_-\right)$, where $\varepsilon = \pm 1$ and $\tau_1 = \frac{\varepsilon + 1}{t} dt$. In addition, M is parallel if and only if $\varepsilon = -1$.*

(ii) *There exists an Einstein locally conformal parallel warped* G_2*-structure on $M = I_f \times L$ with $Scal(g_\varphi) = 42$ if and only if the fiber L belongs to $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$, and then $M = (0, \pi) \times L$ is the $\sin t$-cone with* G_2*-structure $\varphi = \sin^2 t\, \omega \wedge dt + \sin^3 t \left(\varepsilon \frac{\sigma_0}{2}\, \psi_+ - \varepsilon \frac{\pi_0}{2}\, \psi_-\right)$, where $\varepsilon = \pm 1$ and $\tau_1 = \frac{\varepsilon + \cos t}{\sin t} dt$.*

4.2. Einstein coclosed G_2 *manifolds.* In this section we construct Einstein coclosed G_2-structures (i.e. of type $\mathcal{X}_1 \oplus \mathcal{X}_3$) on warped products of $SU(3)$ manifolds in the class $\mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_3$. We apply the construction to the manifold $S^3 \times S^3$ endowed with one of the $SU(3)$-structures described in [41].

Theorem 4.11. *Let $(L, \omega, \psi_+, \psi_-)$ be an Einstein* $SU(3)$*-structure of type $\mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_3$ with $Scal(g_{\omega,\psi_+}) = 30$. Then, the torsion functions π_0, σ_0 are constant, and $C = \sqrt{\pi_0^2 + \sigma_0^2}$ satisfies $C \geq 2$.*

Moreover, let $a = \arccos(\sigma_0/C)$ and consider $\theta(t)$ as follows:

(i) *if $\theta(t)$ is the constant function $\theta = a - \arccos(-2/C)$, then the* G_2*-structure*

$$\varphi = t^2\, \omega \wedge dt + t^3 \Big(\cos\theta\, \psi_+ - \sin\theta\, \psi_- \Big)$$

on the manifold $M = (0, \infty) \times L$ is coclosed and its induced metric is Ricci flat;

(ii) *if* $\theta(t) = a - \arccos(-2\cos t/C)$, *then the* G_2*-structure*

$$\varphi = \sin^2 t\, \omega \wedge dt + \sin^3 t \Big(\cos\theta(t)\, \psi_+ - \sin\theta(t)\, \psi_- \Big)$$

on the manifold $M = (0, \pi) \times L$ *is coclosed and its induced metric is Einstein with* $Scal(g_\varphi) = 42$*;*

(iii) *if* $C > 2$ *and* $\theta(t) = a - \arccos(-2\cosh t/C)$, *then the* G_2*-structure*

$$\varphi = \sinh^2 t\, \omega \wedge dt + \sinh^3 t \Big(\cos\theta(t)\, \psi_+ - \sin\theta(t)\, \psi_- \Big)$$

on the manifold $M = \left(0, \ln \frac{C+\sqrt{C^2-4}}{2}\right) \times L$ *is coclosed, and its induced metric is Einstein with* $Scal(g_\varphi) = -42$.

Proof. Since the SU(3)-structure is of type $\mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_3$, we have that the possibly non-zero torsion reduces to π_0, σ_0 and ν_3, that is, the Eq. (1) reduce to

$$d\omega = -\frac{3}{2}\sigma_0\, \psi_+ + \frac{3}{2}\pi_0\, \psi_- + \nu_3, \qquad d\psi_+ = \pi_0\, \omega^2, \qquad d\psi_- = \sigma_0\, \omega^2.$$

These equations imply $d\pi_0 \wedge \omega^2 = 0$ and $d\sigma_0 \wedge \omega^2 = 0$, therefore the torsion functions π_0, σ_0 are constant.

On the other hand, from the expression (2) for the scalar curvature we get

$$30 = Scal(g_{\omega,\psi_+}) = \frac{15}{2}(\pi_0^2 + \sigma_0^2) - \frac{1}{2}|\nu_3|^2 \leq \frac{15}{2}(\pi_0^2 + \sigma_0^2),$$

which implies $C^2 = \pi_0^2 + \sigma_0^2 \geq 4$.

Moreover, from Corollary 3.5 the G_2-structure given by (7) has torsion form $\tau_2 = 0$. Thus, it is coclosed if and only if $\tau_1 = 0$ or, equivalently by Corollary 3.5, if and only if the equation

$$\sigma_0\, \alpha(t) + \pi_0\, \beta(t) = -2f'(t)$$

is satisfied. The scalar curvature of g_{ω,ψ_+} is positive, so $f(t)$ must be t, $\sin t$ or $\sinh t$.

Let $a = \arccos(\sigma_0/C)$, i.e. $\sigma_0 = C\cos a$ and $\pi_0 = C\sin a$. Writing $\alpha(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$, the equation above becomes $\sigma_0\, \alpha(t) + \pi_0\, \beta(t) = C\cos(a - \theta(t)) = -2f'(t)$, that is,

$$\theta(t) = a - \arccos(-2f'(t)/C).$$

For $f(t) = t$ or $\sin t$, we have that $|-2f'(t)/C| \leq 1$ for any t, because $C \geq 2$. However, for $f(t) = \sinh t$, since $\cosh t \geq 1$ we need to impose that $C > 2$ in order to get an open interval of values of t satisfying $|-2\cosh t/C| < 1$. Indeed, such interval is $(\ln \frac{C-\sqrt{C^2-4}}{2}, \ln \frac{C+\sqrt{C^2-4}}{2})$ when $C > 2$. From this discussion, the cases (i), (ii) and (iii) follow directly. □

Example 4.12. We will apply Theorem 4.11 to an Einstein SU(3)-structure on $S^3 \times S^3$ in the class $\mathcal{W}_1^- \oplus \mathcal{W}_3$ found in [41]. Here we will follow the description given in [37, Section 3.4].

Let us consider the sphere S^3, viewed as the Lie group SU(2), with the basis of left invariant 1-forms $\{e^1, e^2, e^3\}$ satisfying

$$de^1 = e^{23}, \quad de^2 = -e^{13}, \quad \text{and} \quad de^3 = e^{12}.$$

Hence, the Lie algebra of $S^3 \times S^3$ is $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and its structure equations are

$$\mathfrak{g} = (e^{23}, -e^{13}, e^{12}, f^{23}, -f^{13}, f^{12}),$$

where $\{f^i\}$ denotes the basis of 1-forms on the second sphere. Now, we consider the basis $\{h^1, \dots, h^6\}$ of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ given by

$$\begin{aligned} h^1 &= \frac{\sqrt{5}}{10}(e^1 + f^1),\ h^2 = \frac{\sqrt{5}}{10}(-e^1 + f^1),\ h^3 = \frac{\sqrt{10}}{10}\, e^2, \\ h^4 &= \frac{\sqrt{10}}{10}\, f^2,\ h^5 = \frac{\sqrt{10}}{10}\, e^3,\ h^6 = \frac{\sqrt{10}}{10}\, f^3. \end{aligned}$$

With respect to this basis, the structure equations of the Lie algebra $\mathfrak{g}$ of $S^3 \times S^3$ turn into

$$\begin{aligned} \mathfrak{g} = \Big(&\sqrt{5}(h^{35} + h^{46}), \sqrt{5}(-h^{35} + h^{46}), \sqrt{5}(-h^{15} + h^{25}), \\ &\sqrt{5}(-h^{16} - h^{26}), \sqrt{5}(h^{13} - h^{23}), \sqrt{5}(h^{14} + h^{24})\Big). \end{aligned}$$

We define the SU(3)-structure (ω, ψ_+, ψ_-) on $S^3 \times S^3$ by

$$\begin{aligned} \omega &= h^{12} + h^{34} + h^{56}, \qquad \psi_+ = h^{135} - h^{146} - h^{236} - h^{245}, \\ \psi_- &= h^{136} + h^{145} + h^{235} - h^{246}. \end{aligned}$$

Then, an easy calculation shows that the Eq. (1) are

$$\begin{aligned} d\omega &= -\tfrac{3}{2}\sigma_0\, \psi_+ + \nu_3, \\ d\psi_+ &= 0, \\ d\psi_- &= \sigma_0\, \omega \wedge \omega, \end{aligned}$$

where $\sigma_0 = -\sqrt{5}$ and the torsion form ν_3 is given by

$$\nu_3 = -\frac{\sqrt{5}}{2} h^{135} + \frac{\sqrt{5}}{2} h^{146} - \frac{\sqrt{5}}{2} h^{236} - \frac{\sqrt{5}}{2} h^{245} + \sqrt{5}\, h^{235} + \sqrt{5}\, h^{246}.$$

Therefore, the SU(3)-structure (ω, ψ_+, ψ_-) on $S^3 \times S^3$ belongs to the class $\mathcal{W}_1^- \oplus \mathcal{W}_3$. Moreover, the induced metric g_{ω,ψ_+} on $S^3 \times S^3$ is given by $g_{\omega,\psi_+} = \sum_{i=1}^6 h^i \otimes h^i$, and its Ricci curvature tensor satisfies

$$Ric(g_{\omega,\psi_+}) = 5\, g_{\omega,\psi_+}.$$

Thus, g_{ω,ψ_+} is an Einstein metric on $S^3 \times S^3$ with $Scal(g_{\omega,\psi_+}) = 30$.

We can apply Theorem 4.11 to get Einstein coclosed G_2 manifolds with different scalar curvatures. Notice that $C = \sqrt{5}$ and $a = \pi$. Thus, in case (i) we get $\alpha = \frac{2\sqrt{5}}{5}$ and $\beta = -\frac{\sqrt{5}}{5}$, that is, the manifold $M = (0, \infty) \times S^3 \times S^3$ with the G_2-structure

$$\varphi = t^2 \omega \wedge dt + \frac{\sqrt{5}}{5} t^3 \Big(2\, \psi_+ - \psi_-\Big)$$

is a Ricci flat coclosed G_2 manifold.

In case (ii), we have that a slight modification of the sine-cone provides an Einstein coclosed G_2 manifold. More concretely, the G_2-structure

$$\varphi = \sin^2 t\, \omega \wedge dt + \frac{\sqrt{5}}{5} \sin^3 t \Big(2 \cos t\, \psi_+ - \sqrt{5 - 4\cos^2 t}\, \psi_-\Big)$$

on the manifold $M = (0, \pi) \times S^3 \times S^3$ is coclosed and its induced metric is Einstein with positive scalar curvature.

Finally, since $C = \sqrt{5} > 2$ we can apply (iii) with $\theta(t) = \pi - \arccos(-2\cosh t/\sqrt{5})$, to get that the G_2-structure

$$\varphi = \sinh^2 t\, \omega \wedge dt + \frac{\sqrt{5}}{5} \sinh^3 t \Big(2 \cosh t\, \psi_+ - \sqrt{5 - 4\cosh^2 t}\, \psi_-\Big)$$

on the manifold $M = \Big(0, \ln \frac{1+\sqrt{5}}{2}\Big) \times S^3 \times S^3$ is coclosed and its induced metric is Einstein with negative scalar curvature.

4.3. Warped products of Einstein coupled manifolds. In this section we consider warped products of 6-manifolds endowed with a coupled SU(3)-structure. Coupled SU(3)-structures were first introduced in [40] (see also [22] for their role in physics), and they are characterized by the condition

$$d\omega = c\, \psi_+, \tag{17}$$

where $c \in \mathbb{R} - \{0\}$ is a nonzero constant. Equivalently, coupled SU(3)-structures have torsion class $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$, i.e. they are SU(3)-structures for which all the torsion forms different from σ_0 and σ_2 vanish. Notice that the torsion function σ_0 is a constant such that $\sigma_0 = -\frac{2c}{3}$. Coupled SU(3)-structures are half-flat and they generalize the nearly Kähler structures ($\sigma_2 = 0$). The next result follows from Theorem 3.4.

Proposition 4.13. *Let* $(M = I_f \times L, \varphi)$ *be a warped* G_2 *manifold of a coupled* SU(3) *manifold* $(L, \omega, \psi_+, \psi_-)$. *The torsion forms are*

$$\begin{aligned}
\tau_0 &= -\frac{4}{7f}\big(3\,\beta\,\sigma_0 - f\alpha\beta' + f\beta\alpha'\big),\\
\tau_1 &= \frac{1}{2f}(\alpha\,\sigma_0 + 2f')dt,\\
\tau_2 &= -f\alpha\,\sigma_2,\\
\tau_3 &= \frac{3}{14}f^2\left(\alpha\beta\,\sigma_0 + 2f\beta'\right)\psi_+ - \frac{3}{14}f^2\left(\beta^2\sigma_0 - 2f\alpha'\right)\psi_-\\
&\quad - \frac{2}{7}f\left(\beta\,\sigma_0 + 2f\alpha\beta' - 2f\beta\alpha'\right)\omega\wedge dt - f\beta\,\sigma_2\wedge dt,
\end{aligned}$$

where $\sigma_0 = -\frac{2}{3}c$.

Next we will consider coupled SU(3)-structures with $\sigma_2 \neq 0$ (i.e. which are not nearly-Kähler, since the latter case has been studied in Sect. 4.1) which are Einstein with positive scalar curvature. In the following result we restrict our attention to those warped G_2-structures for which α and β are constant.

Theorem 4.14. *Let $(L, \omega, \psi_+, \psi_-)$ be a (non nearly-Kähler) Einstein coupled* SU(3) *manifold with $Scal(g_{\omega,\psi_+}) = 30$. Then, the coupled constant c satisfies $|c| > 3$, and we have:*

(i) *If $(\alpha, \beta) = (1, 0)$, then the* G_2*-structure*

$$\varphi = f^2 \omega \wedge dt + f^3 \psi_+$$

on the manifold $M = I_f \times L$ is locally conformal closed (i.e. of type $\mathcal{X}_2 \oplus \mathcal{X}_4$) and its induced metric is Ricci flat for $f(t) = t$, Einstein with $Scal(g_\varphi) = 42$ for $f(t) = \sin t$, and Einstein with $Scal(g_\varphi) = -42$ for $f(t) = \sinh t$.

(ii) *If $(\alpha, \beta) = (0, 1)$, then the* G_2*-structure*

$$\varphi = f^2 \omega \wedge dt - f^3 \psi_-$$

on the manifold $M = I_f \times L$ is integrable (i.e. of type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$) and its induced metric is Ricci flat for $f(t) = t$, Einstein with $Scal(g_\varphi) = 42$ for $f(t) = \sin t$, and Einstein with $Scal(g_\varphi) = -42$ for $f(t) = \sinh t$.

(iii) *If $(\alpha, \beta) = (\frac{3}{c}, \frac{\sqrt{c^2-9}}{c})$, then the* G_2*-structure*

$$\varphi = t^2 \omega \wedge dt + \frac{t^3}{c}\Big(3\, \psi_+ - \sqrt{c^2-9}\, \psi_-\Big)$$

on the manifold $M = (0, \infty) \times L$ is of type $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ with Ricci flat induced metric.

Proof. Since σ_0, σ_2 do not vanish, from the expression (2) for the scalar curvature we get

$$30 = Scal(g_{\omega,\psi_+}) = \frac{15}{2}\sigma_0^2 - \frac{1}{2}|\sigma_2|^2 = \frac{15}{2}\left(-\frac{2}{3}\,c\right)^2 - \frac{1}{2}|\sigma_2|^2 < \frac{10}{3}\,c^2.$$

Therefore, the coupled constant c in (17) satisfies $c^2 > 9$.

Let α and β be constant functions satisfying $\alpha^2 + \beta^2 = 1$. Then, by Proposition 4.13 the torsion forms of the warped G_2-structure reduce to

$$\tau_0 = \frac{8}{7f}\beta\, c, \qquad \tau_1 = \frac{1}{3f}(3f' - \alpha\, c)dt, \qquad \tau_2 = -f\alpha\,\sigma_2,$$

$$\tau_3 = -\frac{1}{7}f^2\alpha\beta\, c\,\psi_+ + \frac{1}{7}f^2\beta^2 c\,\psi_- + \frac{4}{21}f\beta\, c\,\omega \wedge dt - f\beta\,\sigma_2 \wedge dt,$$

where $\beta = \pm\sqrt{1-\alpha^2}$ and $0 \le |\alpha| \le 1$.

In the case (i), since $\alpha = 1$ and $\beta = 0$ we get

$$\tau_0 = 0, \qquad \tau_1 = \frac{1}{3f}(3f' - c)dt, \qquad \tau_2 = -f\sigma_2, \qquad \tau_3 = 0.$$

Hence the torsion forms τ_0 and τ_3 vanish, i.e. the G_2 manifold is locally conformal closed. Applying Table 3 to the function $f(t) = t$ we have that the induced metric is Ricci flat, and for the function $f(t) = \sin t$ (resp. $f(t) = \sinh t$) the metric induced by the G_2-structure is Einstein with $Scal(g_\varphi) = 42$ (resp. $Scal(g_\varphi) = -42$).

In the case (ii), since $\alpha = 0$ and $\beta = 1$ we have

$$\tau_0 = \frac{8}{7f}c, \quad \tau_1 = \frac{f'}{f}dt, \quad \tau_2 = 0, \quad \tau_3 = \frac{1}{7}f^2 c\,\psi_- + \frac{4}{21}f\,c\,\omega \wedge dt - f\,\sigma_2 \wedge dt.$$

Since $\tau_2 = 0$, the G_2 manifold is of type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$. From Table 3, for the function $f(t) = \sin t$, resp. $f(t) = \sinh t$, the metric induced by the G_2-structure is Einstein with $Scal(g_\varphi) = 42$, resp. $Scal(g_\varphi) = -42$. For $f(t) = t$ the resulting metric is Ricci flat.

In the case (iii), we take $\alpha = 3/c$. Since $|c| > 3$ one has that $|\alpha| < 1$ and we can take β such that $\beta^2 = 1 - \alpha^2 = \frac{c^2-9}{c^2}$. The torsion forms are

$$\tau_0 = \frac{8}{7f}\sqrt{c^2-9}, \qquad \tau_1 = \frac{1}{f}(f'-1)dt, \qquad \tau_2 = -\frac{3}{c}f\,\sigma_2,$$

$$\tau_3 = -\frac{3\sqrt{c^2-9}}{7c}f^2\,\psi_+ + \frac{c^2-9}{7c}f^2\,\psi_- + \frac{4}{21}\sqrt{c^2-9}f\,\omega \wedge dt - \frac{\sqrt{c^2-9}}{c}f\,\sigma_2 \wedge dt.$$

The only possibility for a torsion form to be zero is to consider the function $f(t) = t$ to get $\tau_1 = 0$ (the other torsion forms are clearly non-zero). Therefore, we obtain a G_2 manifold of type $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ with Ricci flat induced metric. □

In order to exemplify this construction we describe first an example of Einstein coupled SU(3)-structure arising from a twistor space.

Example 4.15. It is well-known that the set of positive, orthogonal almost complex structures on a four-dimensional oriented Riemannian manifold forms a smooth manifold $\mathcal{Z}$. The 6-dimensional manifold $\mathcal{Z}$, which is known as the twistor space, admits a (non-integrable) almost complex structure J [17]. If in addition the four-manifold is self-dual Einstein with a suitable positive value of the scalar curvature, then $(\mathcal{Z}, J)$ admits an Einstein coupled SU(3)-structure [43]. Recall that in such case the four-manifold is isometric to the sphere or $\mathbb{CP}^2$ with their canonical metrics (see [5]).

We follow the lines of [22] for the description of this coupled structure. There is a local frame $\{e^1, \ldots, e^6\}$ for the 1-forms on $\mathcal{Z}$ such that the coupled SU(3)-structure (ω, ψ_+, ψ_-) expresses locally as

$$\omega = \frac{8}{5}(e^{12} + e^{34} + e^{56}), \quad \psi_+ = \mathfrak{Re}\,\Psi, \quad \psi_- = \mathfrak{Im}\,\Psi,$$

where

$$\Psi = (8/5)^{\frac{3}{2}}\,i\,(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6).$$

The differential of the forms ω and ψ_- are given by

$$d\omega = -\frac{3}{2}\sigma_0\,\psi_+, \qquad d\psi_- = \sigma_0\,\omega^2 - \sigma_2 \wedge \omega,$$

with

$$\sigma_0 = \frac{\sqrt{10}}{6}(\sigma + 2), \qquad \sigma_2 = -\frac{8\sqrt{10}}{15}(\sigma - 1)\,(e^{12} + e^{34} - 2e^{56}),$$

where $24\,\sigma$ is equal to the scalar curvature of the given four-manifold. The metric induced by the SU(3)-structure is Einstein precisely for the values $\sigma = 1$ (in this case the torsion form σ_2 vanishes and the structure is nearly-Kähler) and $\sigma = 2$. For the latter coupled SU(3)-structure the constant c in (17) is $c = -\sqrt{10}$, and

$$Ric(g_{\omega,\psi_+}) = 5\, g_{\omega,\psi_+},$$

so that we can apply Theorem 4.14.

In the cases (i) and (ii) we get G_2-structures which are locally conformal closed or integrable (i.e. of types $\mathcal{X}_2 \oplus \mathcal{X}_4$ or $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$) whose induced metrics are Ricci flat for $f(t) = t$, Einstein with $Scal(g_\varphi) = 42$ for $f(t) = \sin t$, and Einstein with $Scal(g_\varphi) = -42$ for $f(t) = \sinh t$.

In the case (iii) of Theorem 4.14, since $|c| = \sqrt{10} > 3$, we get that the G_2-structure

$$\varphi = t^2\omega \wedge dt - \frac{t^3}{\sqrt{10}}(3\,\psi_+ - \psi_-),$$

is of type $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ with Ricci flat induced metric.

Remark 4.16. Bryant proved in [9] that there are no closed G_2-structures φ with $Scal(g_\varphi) \geq 0$ unless they are parallel. Indeed, by (6) any such structure satisfies $Scal(g_\varphi) = -\frac{1}{2}|\tau_2|^2$. From Example 4.15 it follows that such a result cannot be extended to the locally conformal closed class, since there are (non parallel) Einstein examples with positive scalar curvature, as well as Ricci flat examples. Notice that the latter case is considered by Fino and Raffero in [22].

In the following result we extend the case (iii) in Theorem 4.14 to more general G_2-structures for which the functions α and β are not constant. This produces new Einstein examples with positive, as well as negative, scalar curvature when we apply the result to a twistor space over a self dual Einstein 4-manifold.

Theorem 4.17. *Let $(L, \omega, \psi_+, \psi_-)$ be a (non nearly-Kähler) Einstein coupled* SU(3) *manifold with $Scal(g_{\omega,\psi_+}) = 30$. Then,*

(i) *the G_2-structure φ on the manifold $M = (0, \pi) \times L$ given by*

$$\varphi = \sin^2 t\, \omega \wedge dt + \frac{\sin^3 t}{c}\left(3\cos t\, \psi_+ - \sqrt{c^2 - 9\cos^2 t}\ \psi_-\right)$$

is of type $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ and its induced metric is Einstein with $Scal(g_\varphi) = 42$;

(ii) *the G_2-structure φ on the manifold $M = \left(0, \ln \frac{|c|+\sqrt{c^2-9}}{3}\right) \times L$ given by*

$$\varphi = \sinh^2 t\, \omega \wedge dt + \frac{\sinh^3 t}{c}\left(3\cosh t\, \psi_+ - \sqrt{c^2 - 9\cosh^2 t}\ \psi_-\right)$$

is of type $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ and its induced metric is Einstein with $Scal(g_\varphi) = -42$.

Proof. By Proposition 4.13 we get that $\tau_1 = 0$ if and only if $\alpha(t) = \frac{3}{c} f'(t)$.

First we consider $f(t) = \sin t$. Since $|c| > 3$ by Theorem 4.14, the function $\alpha(t) = \frac{3}{c}\cos t$ satisfies $|\alpha(t)| < 1$ for any $t \in \mathbb{R}$.

Let us consider now $f(t) = \sinh t$. Since $|c| > 3$, the function $\alpha(t) = \frac{3}{c}\cosh t$ satisfies $|\alpha(t)| \leq 1$ only for the values of $t \in \left[-\ln \frac{|c|+\sqrt{c^2-9}}{3}, \ln \frac{|c|+\sqrt{c^2-9}}{3}\right]$.

Hence, in both cases, the result follows by taking $\beta(t)$ such that $\beta^2(t) = 1 - \alpha^2(t)$. $\square$

Let us consider the twistor space $\mathcal{Z}$ over a self-dual Einstein 4-manifold with the Einstein coupled SU(3)-structure given in Example 4.15. Hence, from Theorem 4.14 (iii) and Theorem 4.17, we obtain G_2 manifolds in the class $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ which are Ricci flat, or Einstein with $Scal(g_\varphi) = \pm 42$.

Einstein G_2 manifolds in the class $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ are given in the following

Theorem 4.18. *Let $(L, \omega, \psi_+, \psi_-)$ be a (non nearly-Kähler) Einstein coupled* SU(3) *manifold with $Scal(g_{\omega,\psi_+}) = 30$. Let c denote the coupled constant, and consider $\theta(t)$ as follows:*

(i) *if $\theta(t) = \arcsin\left(\frac{2t^{-2c}}{1+t^{-4c}}\right)$, then the G_2-structure φ on the manifold $M = (0, \infty) \times L$ given by*

$$\varphi = t^2\omega \wedge dt + t^3(\cos\theta(t)\,\psi_+ - \sin\theta(t)\,\psi_-)$$

belongs to the class $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ and its induced metric is Ricci flat;

(ii) *if $\theta(t) = \arcsin\left(\frac{2(\tan\frac{t}{2})^{-2c}}{1+(\tan\frac{t}{2})^{-4c}}\right)$, then the G_2-structure φ on the manifold $M = (0, \pi) \times L$ given by*

$$\varphi = \sin^2 t\omega \wedge dt + \sin^3 t(\cos\theta(t)\,\psi_+ - \sin\theta(t)\,\psi_-)$$

belongs to the class $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ and its induced metric is Einstein with $Scal(g_\varphi) = 42$;

(iii) *if $\theta(t) = \arcsin\left(\frac{2(\tanh\frac{t}{2})^{-2c}}{1+(\tanh\frac{t}{2})^{-4c}}\right)$, then the G_2-structure φ on the manifold $M = (0, \infty) \times L$ given by*

$$\varphi = \sinh^2 t\omega \wedge dt + \sinh^3 t(\cos\theta(t)\,\psi_+ - \sin\theta(t)\,\psi_-)$$

belongs to the class $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ and its induced metric is Einstein with $Scal(g_\varphi) = -42$.

Proof. Taking $\alpha(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$ in Proposition 4.13 we get that $\tau_0 = \frac{4}{7f}(2c\sin\theta + f\theta')$. A direct calculation shows that for (i), (ii) and (iii) with $f(t) = t$, $\sin t$ and $\sinh t$, respectively, the torsion form τ_0 vanishes, so the G_2-structure belongs to the class $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ and the induced metric is Einstein. Note that τ_1, τ_2 and τ_3 never vanish. □

4.4. Warped products of Einstein solvmanifolds. Up to now, we have constructed Einstein warped G_2 manifolds by means of the warping functions $f(t) = e^t$, $\sinh t$, t or $\sin t$. In view of Table 3, it remains to obtain examples with warping function $f(t) = \cosh t$. Note that in order to obtain such examples, the fiber manifold is required to be Einstein with negative scalar curvature. For this reason, and since Einstein solvmanifolds have negative scalar curvature, in this section we consider the warped products of 6-dimensional solvmanifolds.

An Einstein solvmanifold (S, g) can be described in terms of its Einstein metric solvable Lie algebra, namely $(\mathfrak{s}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$, where $\mathfrak{s}$ is the Lie algebra of the solvable Lie group S, and $\langle\cdot,\cdot\rangle_{\mathfrak{s}}$ is the scalar product on $\mathfrak{s}$. In [35] Lauret obtained a structure theorem for Einstein metric solvable Lie algebras.

Theorem 4.19 [35]. *Any Einstein metric solvable Lie algebra* $(\mathfrak{s}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ *has to be of standard type.*

Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a metric nilpotent Lie algebra. A metric solvable extension of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is a metric solvable Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ such that $\mathfrak{s}$ has the orthogonal decomposition $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ with $[\mathfrak{s},\mathfrak{s}] = \mathfrak{n}$, $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{n}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{s}}|_{\mathfrak{n}\times\mathfrak{n}} = \langle\cdot,\cdot\rangle$. The metric solvable Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ is said to be *standard* or to have *standard type* if $\mathfrak{a}$ is an Abelian subalgebra of $\mathfrak{s}$. In this case, dim $\mathfrak{a}$ is called the *rank*.

Taking into account the structure theorem, in [37, Section 3.2] a classification of Einstein metric 6-dimensional solvable Lie algebras is obtained. There, metric nilpotent Lie algebras up to dimension five are considered, and their corresponding Einstein metric solvable extensions are described.

By considering these 6-dimensional Einstein metric solvable Lie algebras, in the following example we give an Einstein G_2 manifold obtained as a warped product with warping function $f(t) = \cosh t$.

Example 4.20. Let (S, g) be the solvmanifold corresponding to the metric solvable Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ with $\mathfrak{s}$ defined by the structure equations

$$\begin{cases} de^1 = \frac{\sqrt{10}}{4}e^{16}, \\ de^2 = \frac{\sqrt{10}}{4}e^{26}, \\ de^3 = \frac{\sqrt{10}}{4}e^{36}, \\ de^4 = \frac{\sqrt{10}}{4}e^{46}, \\ de^5 = \frac{\sqrt{10}}{2}e^{12} + \frac{\sqrt{10}}{2}e^{34} + \frac{\sqrt{10}}{2}e^{56}, \\ de^6 = 0, \end{cases}$$

and $\langle e^i, e^j\rangle = \delta_{ij}$. Consider the SU(3)-structure (ω, ψ_+, ψ_-) on S given by

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}. \end{aligned}$$

It is clear that the induced metric is precisely the given g, i.e. $g = g_{\omega,\psi_+}$, and it can be checked that

$$Ric(g_{\omega,\psi_+}) = -5\, g_{\omega,\psi_+}.$$

A direct calculation shows that

$$d\omega = 0, \qquad d\psi_+ = \pi_1 \wedge \psi_+, \qquad d\psi_- = \pi_1 \wedge \psi_-,$$

where $\pi_1 = -\sqrt{10}\, e^6$ is the unique non-zero torsion of the SU(3)-structure.

Thus, the SU(3) manifold $(S, \omega, \psi_+, \psi_-)$ is of type $\mathcal{W}_5$ and its induced metric is Einstein with $Scal(g_{\omega,\psi_+}) = -30$. We conclude that the G_2 manifold $(\mathbb{R} \times S, \varphi)$ with

$$\varphi = \cosh^2 t\, \omega \wedge dt + \cosh^3 t\, \psi_+,$$

is of type $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ and its induced metric is Einstein with $Scal(g_\varphi) = -42$. Indeed, by Corollary 3.5 we have $\tau_0 = 0$, and $\tau_1, \tau_2, \tau_3 \neq 0$, because $\pi_1 \neq 0 = \nu_1$.

5. Classification of Einstein G_2-Structures

In this section we apply the results and constructions of Einstein G_2-structures on warped product manifolds given in the previous sections. Motivated by the classification problem studied by Cabrera et al. [12], we realize most of the G_2-classes in the Einstein setting with scalar curvature of different signs. Moreover, at the end of the section we produce several explicit families of Einstein G_2-structures with identical Riemannian metric but having different G_2 type (see [1,9,28,34,36] for related results).

In Table 5 we show concrete Einstein examples, when they exist, in the different Fernández-Gray classes of G_2 manifolds. Since the examples are warped products, in the first column we indicate the fibre. By $\mathcal{NK}$ and $\mathcal{CY}$ we mean a nearly Kähler manifold and a Calabi Yau manifold, respectively. The fiber $S^3 \times S^3$ is the Einstein SU(3) manifold described in Example 4.12. By $\mathcal{Z}$ we mean the twistor space over a self-dual Einstein 4-manifold with the Einstein coupled SU(3)-structure given in Example 4.15. Finally, S is the Einstein solvmanifold given in Example 4.20.

The second, third, and fourth columns give information about the class of the SU(3)-structure on the fiber, the Einstein constant μ of its induced metric, and the torsion forms which are nonzero, respectively.

In Table 5 we also indicate the functions $f(t)$ that give rise to the Einstein G_2 manifolds. The functions $\alpha(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$ defining the appropriate warped G_2-structure in each case are carefully chosen so that the resulting structure provides a strict example in the G_2-class. Here we use the term "strict" to indicate that the G_2-structure does not belong to any subclass of the given one. Next we give details for each G_2-class:

- **The class** $\mathcal{P}$. Examples are given by the t-cone of a nearly Kähler manifold (see Proposition 4.2 and Corollary 4.5).
- **Strict examples in** $\mathcal{X}_1$. Strict examples are given in Proposition 4.3 (see also Corollaries 4.4 and 4.5) as the sine-cone of a nearly Kähler manifold.
- **The classes** $\mathcal{X}_2$ **and** $\mathcal{X}_3$. From Proposition 4.6 (see also Corollary 4.7) one has that via the warped construction it is not possible to obtain strict Einstein examples in these classes.
- **Strict examples in** $\mathcal{X}_4$. Examples are given in Propositions 4.9 and 4.10 as warped products of Calabi-Yau manifolds or, more generally, of Einstein SU(3) manifolds in the class $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$. For instance, for a nearly Kähler manifold, taking $\alpha(t) = 1$ and $\beta(t) = 0$ we get Einstein examples in $\mathcal{X}_4 \backslash \mathcal{P}$ with constant $\lambda = -6$ for $f(t) = \sinh t$, and constant $\lambda = 6$ for $f(t) = \sin t$. Also Ricci flat examples in $\mathcal{X}_4 \backslash \mathcal{P}$ can be obtained with the construction described in Proposition 4.10 (i).
- **The class** $\mathcal{X}_1 \oplus \mathcal{X}_2$. On a connected manifold, one has that $\mathcal{X}_1 \cup \mathcal{X}_2 = \mathcal{X}_1 \oplus \mathcal{X}_2$ (see [12, Theorem 2.1]), so there do not exist strict G_2-structures in this class. From Proposition 4.6 we conclude that there do not exist Einstein warped G_2 manifolds in the class $\mathcal{X}_2$. Thus, the unique Einstein warped G_2 manifolds in the class $\mathcal{X}_1 \oplus \mathcal{X}_2$ are those in $\mathcal{X}_1$.
- **Strict examples in** $\mathcal{X}_1 \oplus \mathcal{X}_3$. The G_2-structures given in Example 4.12 starting from $S^3 \times S^3$ provide Einstein coclosed examples. Moreover, using Corollary 3.5 one can see that the torsion forms $\tau_0, \tau_3 \neq 0$, so they are strict.
- **Strict examples in** $\mathcal{X}_1 \oplus \mathcal{X}_4$. A G_2-structure belongs to $\mathcal{X}_1 \oplus \mathcal{X}_4 \backslash (\mathcal{X}_1 \cup \mathcal{X}_4)$ if and only if the torsion forms satisfy $\tau_2 = \tau_3 = 0$ and $\tau_0, \tau_1 \neq 0$. In order to construct strict examples in the class $\mathcal{X}_1 \oplus \mathcal{X}_4$, we consider a nearly-Kähler manifold L, with torsion $\sigma_0 = -2$ and Einstein constant $\mu = 5$. Let us take $\alpha(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$, with function $\theta(t)$ chosen as follows:

(i) if $\theta(t) = 2\arctan(e^C t)$, with C a constant, and $f(t) = t$, then the corresponding warped G_2-structure on the manifold $(0, \infty) \times L$ belongs to $\mathcal{X}_1 \oplus \mathcal{X}_4 \backslash (\mathcal{X}_1 \cup \mathcal{X}_4)$ and its induced metric is Ricci flat;

(ii) if $\theta(t) = 2\arctan(e^C \tanh \frac{t}{2})$, with C a constant, and $f(t) = \sinh t$, then we get a warped G_2-structure on $(0, \infty) \times L$ sitting in $\mathcal{X}_1 \oplus \mathcal{X}_4 \backslash (\mathcal{X}_1 \cup \mathcal{X}_4)$ whose induced metric is Einstein with $\lambda = -6$;

(iii) if $\theta(t) = 2\arctan(e^C \tan \frac{t}{2})$, with $C \neq 0$ a constant, and $f(t) = \sin t$, then we get a warped G_2-structure on the manifold $(0, \pi) \times L$ that belongs to $\mathcal{X}_1 \oplus \mathcal{X}_4 \backslash (\mathcal{X}_1 \cup \mathcal{X}_4)$ and whose induced metric is Einstein with $\lambda = 6$.

Notice that if in the case (iii) one considers $C = 0$, then one recovers the sine-cone over a nearly-Kähler manifold, and so the G_2-structure belongs to $\mathcal{X}_1 \backslash \mathcal{P}$. For characterization results of manifolds in the strict class $\mathcal{X}_1 \oplus \mathcal{X}_4$, see [14].

- **The class** $\mathcal{X}_2 \oplus \mathcal{X}_3$. By Proposition 4.6 we have that via the warped product construction it is not possible to obtain strict Einstein examples in the class $\mathcal{X}_2 \oplus \mathcal{X}_3$.
- **Strict examples in** $\mathcal{X}_2 \oplus \mathcal{X}_4$. We consider the warped G_2-structures in the class $\mathcal{X}_2 \oplus \mathcal{X}_4$ given in Example 4.15 starting from the twistor space $\mathcal{Z}$ over a self-dual Einstein 4-manifold. Using Corollary 3.5 one can see that the torsion forms $\tau_1, \tau_2 \neq 0$, so they belong to $\mathcal{X}_2 \oplus \mathcal{X}_4 \backslash (\mathcal{X}_2 \cup \mathcal{X}_4)$.
- **Strict examples in** $\mathcal{X}_3 \oplus \mathcal{X}_4$. For strict examples in $\mathcal{X}_3 \oplus \mathcal{X}_4$, we consider the product manifold $S^3 \times S^3$ endowed with the SU(3)-structure given in Example 4.12. Recall that the torsion reduces to $\sigma_0 = -\sqrt{5}$ and $\nu_3 \neq 0$. A G_2-structure belongs to $\mathcal{X}_3 \oplus \mathcal{X}_4 \backslash (\mathcal{X}_3 \cup \mathcal{X}_4)$ if and only if $\tau_0 = \tau_2 = 0$ and $\tau_1, \tau_3 \neq 0$.
  Taking $(\alpha, \beta) = (1, 0)$, we get that the warped G_2-structure $\varphi = f^2 \omega \wedge dt + f^3 \psi_+$ on the manifold $M = I_f \times S^3 \times S^3$ satisfies $\tau_0 = \tau_2 = 0$ and its induced metric is Ricci flat for $f(t) = t$, Einstein with positive scalar curvature for $f(t) = \sin t$, and Einstein with negative scalar curvature for $f(t) = \sinh t$.
  Clearly, $\nu_3 \neq 0$ implies $\tau_3 \neq 0$ by Corollary 3.5. Moreover, $\tau_1 = 0$ if and only if $\sigma_0 \alpha + 2f'(t) = -\sqrt{5} + 2f'(t) = 0$. Hence, it is clear that $\tau_1 \neq 0$ for the functions $f(t) = \sinh t$, t or $\sin t$. In conclusion, one has Einstein examples in $\mathcal{X}_3 \oplus \mathcal{X}_4 \backslash (\mathcal{X}_3 \cup \mathcal{X}_4)$ with Einstein constant $\lambda = -6, 0$ or 6.
- **Strict examples in the classes** $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ **and** $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$. Several strict examples in these classes are constructed in Sect. 4.3 on warped products of Einstein coupled SU(3) manifolds (see Theorems 4.14 (ii)–(iii) and 4.17, and also Example 5.7 below).
- **The class** $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$. This class is the only one where the existence of a strict Einstein warped G_2 manifold remains open. An example could be obtained as follows. Let L be an Einstein SU(3)-structure in the class $\mathcal{W}_1^- \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$, with Einstein constant $\mu = 5$, and such that the nonzero torsion forms satisfy $\sigma_0 = -2$ and $\nu_1 = \pi_1 \neq 0$. The sine-cone of L, i.e. $\alpha(t) = \cos t$ and $\beta(t) = f(t) = \sin t$, would satisfy that $\tau_3 = 0$ and $\tau_0, \tau_1, \tau_2 \neq 0$. However, we do not know of any such L:

Question 5.1. Are there Einstein SU(3)-structures of positive scalar curvature whose nonzero torsion is given by $\sigma_0 = -2$ and $\nu_1 = \pi_1 \neq 0$?

- **Strict examples in** $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$. Einstein examples in this class are given in Theorem 4.18 as a warped product of the twistor space $\mathcal{Z}$, and in Example 4.20 as a warped product of the Einstein solvmanifold S. Since their torsion satisfies that $\tau_0 = 0$ and $\tau_1, \tau_2, \tau_3 \neq 0$, such examples are strict, i.e. they belong to $\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \backslash ((\mathcal{X}_2 \oplus \mathcal{X}_3) \cup (\mathcal{X}_2 \oplus \mathcal{X}_4) \cup (\mathcal{X}_3 \oplus \mathcal{X}_4))$.

- **Strict examples in the general class** $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$. Examples on warped products of the twistor space $\mathcal{Z}$ are given in Example 5.7 below.

We summarize the previous results in the following theorem. By "admissible" we mean that formula (6) does not give an obstruction to the existence of an Einstein G_2-structure with the desired scalar curvature in the given G_2-class.

Theorem 5.2. *For Einstein warped* G_2*-structures, we have:*

(i) *There are Ricci flat warped* G_2*-structures of every admissible strict type, except possibly for* $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$.
(ii) *There are Einstein warped* G_2*-structures with positive scalar curvature of every admissible strict type, except possibly for* $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$.
(iii) *There are Einstein warped* G_2*-structures with negative scalar curvature of every admissible strict type, except for* $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{X}_2 \oplus \mathcal{X}_3$, *and possibly for* $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$.

Motivated by these results, we ask the following general questions:

Question 5.3. Are there Einstein G_2 manifolds in the strict class $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$ with Einstein constant < 0, $= 0$, or > 0?

Question 5.4. Are there Einstein G_2 manifolds with negative scalar curvature in the strict classes $\mathcal{X}_2$, $\mathcal{X}_3$ or $\mathcal{X}_2 \oplus \mathcal{X}_3$?

Remark 5.5. In [15, Section 8.4], cohomogeneity-one metrics are used to construct (Ricci-flat) metrics with holonomy in G_2 and in different admissible G_2-classes. Concerning the class $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$, one can see that the vanishing of the torsion form τ_3 implies that the functions defining the metric must be equal, which leads to $\tau_2 = 0$ and so the G_2-structure lies in $\mathcal{X}_1 \oplus \mathcal{X}_4$.

The results in Sects. 4.2 and 4.3 allow to construct explicit families of G_2-structures in different classes but with the same underlying Einstein metric.

For a fixed Riemannian metric generated by some G_2-structure, it is natural to ask what are the different G_2-structures that induce the same metric. Bryant gave in [9] an answer to this general question, and recently Lin has investigated in [36] the space of parallel G_2-structures inducing the same Riemannian metric on a compact 7-manifold. In the following examples we provide some families of G_2-structures in distinct classes but with identical Einstein metric. We will consider deformations of the form

$$\tilde{\varphi} = \varphi + \chi, \quad \text{where } \chi = f^3(t)\big(A\,\alpha(t)\psi_+ - B\,\beta(t)\psi_-\big)$$

for certain constants A, B. General results on deformations of the form $\tilde{\varphi} = \varphi + \chi$, where χ is a 3-form, are obtained in [28] (see also [34]).

Example 5.6. G_2-structures with identical Einstein metric on warped products of a nearly Kähler manifold. Let us consider L a nearly Kähler manifold and let $f(t) = \sin t$. Following the case (iii) above of strict examples in $\mathcal{X}_1 \oplus \mathcal{X}_4$, consider $\theta_C(t) = 2\arctan(e^C \tan\frac{t}{2})$, where $C \in \mathbb{R}$ is a constant. The G_2-structures φ_C on $M = (0, \pi) \times L$ given by

$$\varphi_C = \sin^2 t\,\omega \wedge dt + \sin^3 t\big(\cos\theta_C(t)\,\psi_+ - \sin\theta_C(t)\,\psi_-\big)$$

satisfy that $g_{\varphi_C} = dt^2 + \sin^2 t\, g_L$, i.e. the induced Einstein metric is identical for all the G_2-structures in the family. The G_2-type of φ_C varies as follows:

- $\mathcal{X}_1$, if and only if $C = 0$;

- $\mathcal{X}_1 \oplus \mathcal{X}_4$, if and only if $C \neq 0$.

Therefore, we can deform the structure in $\mathcal{X}_1$ to one in the class $\mathcal{X}_1 \oplus \mathcal{X}_4$.

Example 5.7. G_2-structures with identical Einstein metric on warped products of the twistor space $\mathcal{Z}$. We define an explicit family of G_2-structures in different classes but with the same induced Ricci flat metric starting from L in the conditions of Theorem 4.14 (iii), in particular for $L = \mathcal{Z}$. Let us denote $\alpha_0 = \frac{3}{c}$ and $\beta_0 = \frac{\sqrt{c^2-9}}{c}$, and consider $(a, b) \in \mathbb{R}^2$ the points in the ellipse of equation $\alpha_0^2\, a^2 + \beta_0^2\, b^2 = 1$. On $M = (0, \infty) \times L$ we take the family of G_2-structures

$$\varphi_{a,b} = t^2 \omega \wedge dt + t^3\, (a\, \alpha_0\, \psi_+ - b\, \beta_0\, \psi_-)\,.$$

The induced Ricci flat metric is $g_{\varphi_{a,b}} = dt^2 + t^2 g_L$, but the G_2-structure belongs to the strict class

- $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$, if and only if $(a, b) = (1, 1)$;
- $\mathcal{X}_2 \oplus \mathcal{X}_4$, if and only if $(a, b) = (\alpha_0^{-1}, 0)$;
- $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$, if and only if $(a, b) = (0, \beta_0^{-1})$;
- $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ for any other values of (a, b).

Similar families can be constructed for the other Einstein metrics based on $f(t) = \sin t$ and $f(t) = \sinh t$. Take $(a, b) \in \mathbb{R}^2$ satisfying $a^2 + b^2 = 1$. On $M = (0, \infty) \times L$, we consider the family of G_2-structures

$$\varphi_{a,b} = f^2 \omega \wedge dt + f^3\, (a\, \psi_+ - b\, \psi_-)\,.$$

The induced Einstein metric is $g_{\varphi_{a,b}} = dt^2 + f^2 g_L$, and by Theorem 4.14 (i) (ii), the G_2-structure belongs to the strict class

- $\mathcal{X}_2 \oplus \mathcal{X}_4$, if and only if $(a, b) = (1, 0)$;
- $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$, if and only if $(a, b) = (0, 1)$;
- $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ for any other values of (a, b).

The Einstein constant is positive, resp. negative, for $f(t) = \sin t$, resp. $f(t) = \sinh t$.

6. Spin(7)-Structures

In this section we consider Spin(7) manifolds given as a warped product of a G_2 manifold, and we obtain an explicit description of the torsion forms of the warped Spin(7)-structure in terms of the torsion forms of the G_2-structure.

A Spin(7)-structure on an 8-dimensional manifold N consists of a reduction of the structure group of its frame bundle to the Lie group Spin(7). Equivalently, such structure can be characterized by the existence of a global non-degenerate 4-form ϕ on N which can be locally written as

$$\begin{aligned} \phi = \;& e^{1278} + e^{3478} + e^{5678} + e^{1358} - e^{1468} - e^{2368} - e^{2458} \\ & + e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467}, \end{aligned} \tag{18}$$

where $\{e^1, \ldots, e^8\}$ is a local basis of 1-forms on N. The presence of a Spin(7)-structure ϕ on a manifold defines a volume form vol_8 and a Riemannian metric g_ϕ which satisfy

$$\big(g_\phi(X, X) g_\phi(Y, Y) - g_\phi(X, Y)^2\big) vol_8 = \frac{1}{6} \iota_X \iota_Y \phi \wedge \iota_X \iota_Y \phi \wedge \phi,$$

where $g_\phi(U, U)$ is given explicitly in [34, Corollary 4.3.2].

Given a Spin(7) manifold (N, ϕ), the group Spin(7) acts on the space of differential p-forms $\Omega^p(N)$ on N. This action is irreducible on $\Omega^1(N)$ and $\Omega^7(N)$, but it is reducible for $\Omega^p(N)$ with $2 \le p \le 6$. Since the Hodge star operator $*_8$ induces an isomorphism $*_8\, \Omega^p(N) \cong \Omega^{8-p}(N)$, it suffices to describe the decompositions for $p = 2, 3$ and 4. In [8] it is shown that the Spin(7) irreducible decompositions for $2 \le p \le 4$ are

$$\Omega^2(N) = \Omega^2_7(N) \oplus \Omega^2_{21}(N),$$
$$\Omega^3(N) = \Omega^3_8(N) \oplus \Omega^3_{48}(N),$$
$$\Omega^4(N) = \Omega^4_1(N) \oplus \Omega^4_7(N) \oplus \Omega^4_{27}(N) \oplus \Omega^4_{35}(N),$$

where $\Omega^p_k(N)$ denotes the Spin(7) irreducible space of p-forms of dimension k at every point. The description on the other degrees is obtained via the isomorphism $*_8\, \Omega^p_k(N) \cong \Omega^{8-p}_k(N)$ given by the Hodge star operator, and in this section we are only interested in the Spin(7)-type decomposition of 5-forms. This space decomposes as

$$\Omega^5(N) = \Omega^5_8(N) \oplus \Omega^5_{48}(N),$$

where

$$\Omega^5_8(N) = \{\alpha \wedge \phi \mid \alpha \in \Omega^1(N)\},$$
$$\Omega^5_{48}(N) = \{\gamma \in \Omega^5(N) \mid \phi \wedge *_8\gamma = 0\}.$$

The isomorphisms between Spin(7) irreducible spaces introduce a scaling factor on 1-forms $\kappa \in \Omega^1(N)$ as follows:

$$*_8\,(*_8(\kappa \wedge \phi) \wedge \phi) = -7\kappa. \tag{19}$$

The above decomposition of 5-forms on N allows to express the exterior derivative of ϕ as

$$d\phi = \lambda_1 \wedge \phi + \lambda_5, \tag{20}$$

where $\lambda_1 \in \Omega^1(N)$ and $\lambda_5 \in \Omega^5_{48}(N)$ are called the *torsion forms* of the Spin(7)-structure.

According to [18] the covariant derivative of ϕ can be decomposed into two components, namely Y_1 and Y_2. Thus, a Spin(7)-structure is said of type $\mathcal{P}, \mathcal{Y}_1, \mathcal{Y}_2$ or $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ if the covariant derivative $\nabla^{g_\phi}\phi$ lies in $\{0\}, Y_1, Y_2$ or $Y = Y_1 \oplus Y_2$, respectively. In terms of the torsion forms, these classes are characterized in Table 4. In the parallel case, the holonomy reduces to Spin(7) and the metric is Ricci-flat. Examples of manifolds with Spin(7) holonomy are constructed in [9,10,33].

Table 4. Classes of Spin(7)-structures

Class	Torsion forms	Structure
$\mathcal{P}$	$\lambda_1 = \lambda_5 = 0$	Parallel
$\mathcal{Y}_1$	$\lambda_5 = 0$	Locally conformal parallel
$\mathcal{Y}_2$	$\lambda_1 = 0$	Balanced
$\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$	No condition	General Spin(7)

As it happened for SU(3) and G$_2$ manifolds, the scalar curvature of a Spin(7) manifold can be described in terms of the torsion forms. The expression can be achieved from the formulas described in [31,39] and is given as follows:

$$Scal(g_\phi) = \frac{21}{8}|\lambda_1|^2 - \frac{1}{2}|\lambda_5|^2 + \frac{7}{2}d^{*_8}\lambda_1, \tag{21}$$

where d^{*_8} denotes the codifferential, i.e. the adjoint operator of the exterior derivative with respect to the metric.

Consider a 7-dimensional manifold M endowed with a G$_2$-structure φ. Let N be the Riemannian product $N = \mathbb{R} \times M$, and denote by $p\colon N \longrightarrow \mathbb{R}$ and $q\colon N \longrightarrow M$ the projections. Then, the 4-form

$$\phi = q^*(\varphi) \wedge p^*(dt) + q^*(*_7\varphi),$$

with t the coordinate on $\mathbb{R}$, defines a Spin(7)-structure on N. In the following, φ and $*_7\varphi$ will be identified with their pullbacks onto N. More generally, we have

Proposition 6.1. *Let* (M, φ) *be a* G$_2$ *manifold and consider a function* $f\colon I_f \longrightarrow \mathbb{R}$. *Then, the* 4*-form on* $N = I_f \times M$ *given by*

$$\phi = f^3(t)\,\varphi \wedge dt + f^4(t)\,*_7\varphi \tag{22}$$

defines a Spin(7)*-structure with induced metric*

$$g_\phi = f^2(t)\,g_\varphi + dt^2,$$

and volume form $vol_8 = f^7(t)vol_7 \wedge dt$.

Proof. Let $\{e^1, \dots, e^7\}$ be a local orthonormal basis of 1-forms such that the 3-form φ writes as in (3). Now, with respect to the local basis on N given by $\{h^1, \dots, h^8\} = \{f(t)e^1, \dots, f(t)e^7, dt\}$, the 4-form ϕ can be written as in (18). Therefore, $\{h^1, \dots, h^8\}$ is orthonormal for the metric g_ϕ, and

$$g_\phi = \sum_{i=1}^{8} h^i \otimes h^i = f^2(t)\sum_{i=1}^{7} e^i \otimes e^i + dt \otimes dt = f^2(t)\,g_\varphi + dt^2.$$

□

By the preceding proposition, the Spin(7) manifold $N = I_f \times M$ with ϕ described in (22) corresponds, as a Riemannian manifold, to the warped product $N = I_f \times_f M$. We will refer to such a Spin(7)-structure as a *warped* Spin(7)*-structure*, and the manifold $(N = I_f \times M, \phi)$ will be called *warped* Spin(7) *manifold*.

Lemma 6.2. *Let* $\beta \in \Omega^q(M)$ *be a differential q-form on M, and let* $*_7$ *and* $*_8$ *be the Hodge star operators induced by the structures* φ *and* ϕ*, respectively. Then,*

$$*_8\beta = f^{7-2q} *_7\beta \wedge dt, \qquad *_8(\beta \wedge dt) = (-1)^{q+1} f^{7-2q} *_7\beta.$$

Proof. It is a consequence of the fact that the Hodge star operator $*_8$ is determined by (g_ϕ, vol_8), where $vol_8 = f^7 vol_7 \wedge dt$ and $vol_7 = \frac{1}{7}\varphi \wedge *_7\varphi$. □

Theorem 6.3. *Let* (M, φ) *be a* G_2 *manifold with torsion forms* $\tau_0, \tau_1, \tau_2, \tau_3$. *Then, the torsion forms* λ_1, λ_5 *of a warped* Spin(7) *manifold* $(N = I_f \times M, \phi)$ *are given by*

$$\lambda_1 = \frac{1}{f}(\tau_0 + 4f')\, dt + \frac{24}{7}\tau_1,$$
$$\lambda_5 = -\frac{3}{7} f^3\, \tau_1 \wedge \varphi \wedge dt + \frac{4}{7} f^4\, \tau_1 \wedge *_7\varphi + f^4\, \tau_2 \wedge \varphi + f^3 *_7\tau_3 \wedge dt.$$

Proof. From (19) and (20), and since $\lambda_5 \in \Omega^5_{48}(N)$, it follows that the torsion form λ_1 is given by

$$\lambda_1 = -\frac{1}{7} *_8 \big((*_8 d\,\phi) \wedge \phi\big).$$

In order to compute $*_8 d\phi$, we first take into account (5) and (22) to get

$$\begin{aligned} d\phi = {}& f^3(\tau_0 + 4f')\ *_7\varphi \wedge dt + 3f^3\, \tau_1 \wedge \varphi \wedge dt + f^3 *_7\tau_3 \wedge dt \\ & +4f^4\, \tau_1 \wedge *_7\varphi + f^4\, \tau_2 \wedge \varphi. \end{aligned}$$

A direct calculation using Lemma 6.2 shows that

$$\begin{aligned} *_8 d\phi = {}& -f^2(\tau_0 + 4f')\,\varphi - 3f^2 *_7(\tau_1 \wedge \varphi) - f^2\tau_3 + 4f *_7(\tau_1 \wedge *_7\varphi) \wedge dt \\ & + f *_7(\tau_2 \wedge \varphi) \wedge dt. \end{aligned}$$

Now, by (4) we arrive at

$$\begin{aligned} (*_8 d\phi) \wedge \phi = {}& -f^6(\tau_0 + 4f')\,\varphi \wedge *_7\varphi \\ & - 3f^5 *_7(\tau_1 \wedge \varphi) \wedge \varphi \wedge dt - 3f^6 *_7(\tau_1 \wedge \varphi) \wedge *_7\varphi \\ & + 4f^5 *_7(\tau_1 \wedge *_7\varphi) \wedge *_7\varphi \wedge dt \\ = {}& -f^6(\tau_0 + 4f')\,\varphi \wedge *_7\varphi + 24\, f^5 *_7\tau_1 \wedge dt. \end{aligned}$$

Then, using again Lemma 6.2, we get

$$*_8\big((*_8 d\phi) \wedge \phi\big) = -\frac{7}{f}(\tau_0 + 4f')\, dt - 24\,\tau_1,$$

concluding that

$$\lambda_1 = \frac{1}{f}(\tau_0 + 4f')\, dt + \frac{24}{7}\tau_1.$$

Finally, for the torsion form λ_5 we use that $\lambda_5 = d\phi - \lambda_1 \wedge \phi$, together with the expressions of $d\phi$ and λ_1 given above. □

A direct consequence of the previous theorem is the following

Corollary 6.4. *The torsion forms of a warped* Spin(7)*-structure satisfy:*

$$\lambda_1 = 0 \iff \begin{cases} i) & \tau_0 + 4f' = 0, \\ ii) & \tau_1 = 0. \end{cases}$$

$$\lambda_5 = 0 \iff \begin{cases} iii) & \tau_1 = 0, \\ iv) & \tau_2 = 0, \\ v) & \tau_3 = 0. \end{cases}$$

7. Einstein Warped Spin(7) Manifolds

Our aim in this section is to construct Einstein 8-manifolds in the different Spin(7)-classes by means of warped products of certain Einstein G_2 manifolds, i.e. by means of warped Spin(7)-structures. As in Sect. 4, in order to use directly Table 3, in this section we will also consider the Einstein metrics to be "normalized".

We begin with a characterization of the warped Spin(7) manifolds that are parallel, which is related to a well known result in [3].

Proposition 7.1. *There exists a parallel warped* Spin(7)*-structure on* $N = I_f \times M$ *if and only if the fiber* (M, φ) *belongs to* $\mathcal{X}_1$*, i.e. it is a nearly parallel* G_2 *manifold, with torsion* $\tau_0 = -4$.

Furthermore, in that case $N = (0, \infty) \times M$ *is the cone with* Spin(7)*-structure*

$$\phi = t^3 \varphi \wedge dt + t^4 *_7 \varphi.$$

Proof. The parallel condition on the Spin(7)-structure is equivalent to $\lambda_1 = \lambda_5 = 0$. From Corollary 6.4, and taking into account the possible functions in Table 3, these equations are equivalent to

$$\tau_1 = \tau_2 = \tau_3 = 0, \qquad \tau_0 = -4 \qquad \text{and} \qquad f(t) = t,$$

and the result follows. □

The following three propositions give characterizations of the warped Spin(7) manifolds that are Einstein and locally conformal parallel, depending on the sign of its scalar curvature.

Proposition 7.2. *There exists an Einstein locally conformal parallel warped* Spin(7)*-structure* ϕ *on* $N = I_f \times M$ *with* $Scal(g_\phi) = 56$ *if and only if the fiber* (M, φ) *belongs to* $\mathcal{X}_1$ *with torsion* $\tau_0 = \pm 4$.

Furthermore, in that case $N = (0, \pi) \times M$ *is the sine-cone with* Spin(7)*-structure*

$$\phi = \sin^3 t\, \varphi \wedge dt + \sin^4 t\, *_7 \varphi.$$

Proof. Suppose there exists such a warped product $(N = I_f \times M, \phi)$. Since $\lambda_5 = 0$, Corollary 6.4 forces the G_2-structure φ to be in the class $\mathcal{X}_1$. Since $Scal(g_\phi) = 56$, by Table 3 we get that the warping function is necessarily given by $f(t) = \sin t$ and $Scal(g_\varphi) = 42$. Now, by (6), the torsion of the G_2-structure is $\tau_0 = \pm 4$.

Conversely, if we consider a nearly parallel G_2 manifold with torsion $\tau_0 = \pm 4$, then the warped Spin(7)-structure with $f(t) = \sin t$ is Einstein (with constant 7) and locally conformal parallel by Corollary 6.4. □

Proposition 7.3. *There exists a Ricci flat (strict) locally conformal parallel warped* Spin(7)*-structure* ϕ *on* $N = I_f \times M$ *if and only if the fiber* (M, φ) *belongs to* $\mathcal{X}_1$ *with torsion* $\tau_0 = 4$.

Furthermore, in that case $N = (0, \infty) \times M$ *is the cone with* Spin(7)*-structure*

$$\phi = t^3 \varphi \wedge dt + t^4 *_7 \varphi.$$

Proof. The proof is similar to that of Proposition 7.2, but taking into account that the Ricci flatness forces the warping function to be $f(t) = t$. Hence, the locally conformal parallel Spin(7)-structure is strict only when $\tau_0 = 4$. □

Proposition 7.4. *There exists an Einstein locally conformal parallel warped* Spin(7)*-structure ϕ on $N = I_f \times M$ with $Scal(g_\phi) = -56$ if and only if the* G_2*-structure φ on the fiber M is one of the following:*

- *Parallel, and then $N = \mathbb{R} \times M$ is the exponential-cone with the* Spin(7)*-structure*

$$\phi = e^{3t}\,\varphi \wedge dt + e^{4t} *_7 \varphi;$$

- *Nearly parallel with torsion $\tau_0 = \pm 4$, and then $N = (0, \infty) \times M$ is the hyperbolic sine-cone with the* Spin(7)*-structure*

$$\phi = \sinh^3 t\,\varphi \wedge dt + \sinh^4 t\ *_7 \varphi.$$

Proof. The proof is similar to the preceding propositions, but since $Scal(g_\phi) = -56$, by Table 3 we have that either $\tau_0 = 0$ and $f(t) = e^t$, or $Scal(g_\varphi) = 42$ and $f(t) = \sinh t$. In the first case the fiber is parallel, and in the second case it is a nearly parallel G_2 manifold with torsion $\tau_0 = \pm 4$. □

As a consequence one gets Einstein locally conformal parallel Spin(7) manifolds with negative, zero or positive constant (see Corollary 4.4 for G_2 manifolds satisfying the hypothesis of the following corollary) (Table 5).

Corollary 7.5. *Let (M, φ) be a nearly parallel* G_2 *manifold with torsion $\tau_0 = 4$. Then, there are warped* Spin(7)*-structures with fiber (M, φ) which are (strict) locally conformal parallel and Einstein with constant -7, 0 or 7, by taking the function $f(t) = \sinh t$, t or $\sin t$, respectively.*

In the following result we note that there are no Einstein (strict) balanced warped Spin(7) manifolds.

Proposition 7.6. *A warped* Spin(7) *manifold is balanced and Einstein if and only if it is a parallel* Spin(7) *manifold.*

Proof. Given an Einstein balanced warped Spin(7) manifold, since $\lambda_1 = 0$, from Corollary 6.4 we get that the torsion forms of the G_2-structure on the fiber satisfy

$$\tau_1 = 0, \qquad \tau_0 = -4$$

and the warping function in Table 3 is $f(t) = t$. Thus, the Spin(7)-structure is necessarily Ricci flat and by (21) we get $\lambda_5 = 0$. In conclusion, the warped Spin(7)-structure is parallel. □

As in Sect. 5, we summarize in Table 6 the results obtained above for Einstein warped Spin(7) manifold in the different strict classes:

- **The class $\mathcal{P}$**. Examples are given by the t-cone of a nearly parallel G_2 manifold (see Proposition 7.1).
- **The class $\mathcal{Y}_1$**. Strict examples with Einstein constant -7, 0 or 7 are given in Corollary 7.5 as the hyperbolic sine-cone, cone or sine-cone, respectively, of a nearly parallel G_2 manifold with torsion $\tau_0 = 4$.
- **The class $\mathcal{Y}_2$**. By Proposition 7.6 it is not possible to obtain strict Einstein examples via the warped construction.
- **The general class $\mathcal{Y}_1 \oplus \mathcal{Y}_2$**. Strict examples with positive, null and negative scalar curvature can be achieved as the different cones of Einstein locally conformal parallel G_2 manifolds (see Sect. 5 for examples of such G_2 manifolds).

Table 5. Einstein warped G_2-structures

Fiber	SU(3)-class	μ	SU(3)non-vanishing torsion forms	$f(t)$-cone metric	Strict G_2-class	Einstein constant λ	G_2 non-vanishing torsion forms
$\mathcal{NK}$	$\mathcal{W}_1^-$	5	$\sigma_0 = -2$	t	$\mathcal{P}$	0	–
$\mathcal{NK}$	$\mathcal{W}_1^-$	5	$\sigma_0 = -2$	$\sin t$	$\mathcal{X}_1$	6	τ_0
$\nexists$					$\mathcal{X}_2$		
$\nexists$					$\mathcal{X}_3$		
$\mathcal{CY}$	$\{0\}$	0	–	e^t	$\mathcal{X}_4$	-6	τ_1
$\mathcal{NK}$	$\mathcal{W}_1^-$	5	$\sigma_0 = -2$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_4$	$-6,\ 0,\ 6$	τ_1
$\nexists$					$\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{X}_1 \cup \mathcal{X}_2$		
$S^3 \times S^3$	$\mathcal{W}_1^- \oplus \mathcal{W}_3$	5	$\sigma_0,\ \nu_3$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_1 \oplus \mathcal{X}_3$	$-6,\ 0,\ 6$	$\tau_0,\ \tau_3$
$\mathcal{NK}$	$\mathcal{W}_1^-$	5	$\sigma_0 = -2$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_1 \oplus \mathcal{X}_4$	$-6,\ 0,\ 6$	$\tau_0,\ \tau_1$
$\nexists$					$\mathcal{X}_2 \oplus \mathcal{X}_3$		
$\mathcal{Z}$	$\mathcal{W}_1^- \oplus \mathcal{W}_2^-$	5	$\sigma_0,\ \sigma_2$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_2 \oplus \mathcal{X}_4$	$-6,\ 0,\ 6$	$\tau_1,\ \tau_2$
$S^3 \times S^3$	$\mathcal{W}_1^- \oplus \mathcal{W}_3$	5	$\sigma_0,\ \nu_3$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_3 \oplus \mathcal{X}_4$	$-6,\ 0,\ 6$	$\tau_1,\ \tau_3$
$\mathcal{Z}$	$\mathcal{W}_1^- \oplus \mathcal{W}_2^-$	5	$\sigma_0,\ \sigma_2$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$	$-6,\ 0,\ 6$	$\tau_0,\ \tau_2,\ \tau_3$
(?)					$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_4$		$\tau_0,\ \tau_1,\ \tau_2$
$\mathcal{Z}$	$\mathcal{W}_1^- \oplus \mathcal{W}_2^-$	5	$\sigma_0,\ \sigma_2$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	$-6,\ 0,\ 6$	$\tau_0,\ \tau_1,\ \tau_3$
$\mathcal{S}$	$\mathcal{W}_5$	-5	π_1	$\cosh t$	$\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	-6	$\tau_1,\ \tau_2,\ \tau_3$
$\mathcal{Z}$	$\mathcal{W}_1^- \oplus \mathcal{W}_2^-$	5	$\sigma_0,\ \sigma_2$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	$-6,\ 0,\ 6$	$\tau_1,\ \tau_2,\ \tau_3$
$\mathcal{Z}$	$\mathcal{W}_1^- \oplus \mathcal{W}_2^-$	5	$\sigma_0,\ \sigma_2$	$\sinh t,\ t,\ \sin t$	$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	$-6,\ 0,\ 6$	$\tau_0,\ \tau_1,\ \tau_2,\ \tau_3$

Table 6. Einstein warped Spin(7)-structures

Fiber	G_2-class	μ	G_2 non-vanishing torsion forms	$f(t)$-cone metric	Strict Spin(7)-class	Einstein constant λ	Spin(7) non-vanishing torsion forms
$\mathcal{NP}$	$\mathcal{X}_1$	6	$\tau_0 = -4$	t	$\mathcal{P}$	0	–
$\mathcal{NP}$	$\mathcal{X}_1$	6	τ_0	$\sinh t,\ t,\ \sin t$	$\mathcal{Y}_1$	$-7,\ 0,\ 7$	λ_1
$\mathcal{P}$	$\{0\}$	0	–	e^t	$\mathcal{Y}_1$	-7	λ_1
$\nexists$					$\mathcal{Y}_2$		
$\mathcal{LCP}$	$\mathcal{X}_4$	6	τ_1	$\sinh t,\ t,\ \sin t$	$\mathcal{Y}_1 \oplus \mathcal{Y}_2$	$-7,\ 0,\ 7$	$\lambda_1,\ \lambda_5$
$\mathcal{LCP}$	$\mathcal{X}_4$	0	τ_1	e^t	$\mathcal{Y}_1 \oplus \mathcal{Y}_2$	-7	$\lambda_1,\ \lambda_5$
$\mathcal{LCP}$	$\mathcal{X}_4$	-6	τ_1	$\cosh t$	$\mathcal{Y}_1 \oplus \mathcal{Y}_2$	-7	$\lambda_1,\ \lambda_5$

We summarize the previous results in the following

Theorem 7.7. *For Einstein warped* Spin(7)*-structures, we have:*

(i) *There are Ricci flat warped* Spin(7)*-structures of every admissible strict type.*
(ii) *There are Einstein warped* Spin(7)*-structures with positive scalar curvature of every admissible strict type.*
(iii) *There are Einstein warped* Spin(7)*-structures with negative scalar curvature of every admissible strict type, except for* $\mathcal{Y}_2$.

Motivated by this result, we ask the following question:

Question 7.8. Are there Einstein (non parallel) balanced Spin(7) manifolds?

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